Study of a family of stationary and axially symmetric differentially rotating perfect fluids

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The geometrical analysis and interpretation of a family of stationary and axisymmetric differentially rotating perfect fluids is performed. The family was first found by Senovilla and later by Mars and Senovilla under completely different hypotheses, and contains an arbitrary function of a single coordinate so there exists a large variety of behaviors. We find under which conditions the spacetime is axially symmetric with a well-defined axis of symmetry and when it satisfies the energy conditions. By imposing the reasonable assumption that the rotating body is finite in size we find two geometrically different classes of spacetimes. One of them represents a topological three-sphere with special properties. The other class is very interesting from the physical point of view, as it typically represents an isolated compact body with axial symmetry (with or without equatorial symmetry), and possible pointlike singularities at the north and south poles. There are even exotic examples in this class, as these objects can contain central and/or toroidal empty holes. As far as we know, these models are the more realistic explicit solutions produced so far by Einstein's theory in order to describe the interior of axially symmetric differentially rotating isolated compact bodies. [S0556-2821(96)04222-1]

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I. INTRODUCTION

There exist very few examples of explicit spacetimes describing the geometry (and thus the gravitational field) inside self-gravitating rotating bodies. Even in the equilibrium case when the metric can be assumed to be stationary and restricting the shape of the body to be axially symmetric, the number of explicit nonvacuum solutions is indeed very small. Furthermore, to understand the behavior of rotating self-gravitating bodies we have to face another major problem (in addition to the lack of explicit solutions): the lack of knowledge of the exterior spacetime (usually vacuum) matching properly with the given interior. Only when this exterior is known can we give a full account of the properties of the rotating body. However, this important question has not been answered yet, neither from a theoretical point of view (an existence and uniqueness theorem of the exterior given the interior) nor from a practical point of view since *no* explicit global spacetime together with its interior is explicitly known (apart from static spherically symmetric cases and spacetimes with cylindrical symmetry; see, e.g., [1] and references therein). Until this existence problem is addressed and global solutions (for compact rotating bodies) are found, we have to be content with an analysis of the interior metric, whenever this is possible, keeping in mind that this kind of study is necessarily incomplete and somewhat speculative because the interpretation of the interior solution must be taken as tentative rather than definite until an exterior solution (if any) is found.

Nevertheless, this does not mean that we can say nothing about the interior body under consideration. For instance, we can undoubtedly assert under which circumstances the interior metric has well-defined axial symmetry, which type of matter content is filling the rotating body, whether or not this matter satisfies energy conditions, the shape and properties of the limit surface (if it exists) where the body ends and the exterior solution should be matched, etc. If we are dealing with a family of interiors rather than with a single solution, we can also perform a detailed study of the rigidly rotating (including static) subcases and of the particular solutions with a larger isometry group, which can give us some insight into the interpretation of the general family. Finally, we can also try to determine the properties and shapes of the interior from the local metric tensor via some geometrical interpretation of the coordinates. Thus, we see that the analysis of the interior solution is still interesting and necessary (much more so given the lack of explicit examples at hand). In this paper we shall perform one of these geometrical analyses for a rather large family of stationary and axisymmetric perfect-fluid solutions. This family was first found by one of us [2] as the most general stationary and axisymmetric nonconvective and differentially rotating perfect-fluid solution satisfying the assumptions of being Petrov type D with the fluid velocity vector lying in the two-planes spanned by the two principal null directions, and with vanishing magnetic part of the Weyl tensor with respect to the fluid velocity vector. The family was later refound under completely different hypotheses [3,4]: It is the most general differentially rotating stationary and axially symmetric nonconvective perfect fluid admitting one *proper* (nonhomothetic) conformal motion.

The plan of the paper is as follows. In Sec. II we perform a purely geometrical analysis of the line element without imposing the perfect-fluid Einstein field equations. We give the most general coordinate transformation which leaves the form of the line element invariant and find under which con-

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ditions the spacetime is indeed axially symmetric with a well-defined axis of symmetry. It is shown that in some situations closed timelike curves (thus violating causality) will occur. This allows us to restrict the physically interesting situations. In Sec. III we write down and partially solve the Einstein field equations, from where we see that the general perfect-fluid solution depends on an arbitrary function of one coordinate. The remaining differential equations are simple and can be seen as the equations of motion of a particle in a three-dimensional Minkowski spacetime. In Sec. IV, we study the matter content of the spacetime and impose the energy conditions everywhere inside the body, thereby further restricting the physically relevant solutions contained in the family. In this section we also give the limit hypersurface of the interior body explicitly. In Sec. V we find all the rigidly rotating solutions contained in the family. There exist two different rigidly rotating subfamilies. One has a two-dimensional isometry group in general (apart from a particular case which reduces to the de Sitter spacetime) and the other has a four-dimensional isometry group. The geometry of this second family can be easily interpreted, and this gives us some ideas for the analysis of the general case. Then, in Sec. VI, we classify the different possible behaviors of the solutions and prove that there exist two physically well-behaved classes of solutions. The first represents a topological three-sphere with two singularities at the north and south poles and with a hole symmetrically distributed around the equatorial plane. The other class is much more interesting and we devote the whole of Sec. VII to its study. In this last class, there appear all types of possible shapes for the interior body: Typically, they represent an isolated compact body with axial and equatorial symmetry, either prolatum, oblatum, or irregular, and two pointlike singularities at the north and south poles. However, there are even more exotic examples, as these objects can contain central and/or toroidal empty holes. Finally, there are also cases of axially symmetric compact objects *without* equatorial symmetry. As far as we know, the models presented in Sec. VII are the best and more realistic explicit models produced so far by Einstein's theory in order to describe the interior of axially symmetric differentially rotating isolated compact bodies.

II. GEOMETRICAL ANALYSIS OF THE LINE ELEMENT

As stated above, the subject of our study is the general solution for stationary and axisymmetric differentially rotating non-convective perfect fluids admitting a proper conformal motion. As shown in [2] and [3] the line element can be written in the form

$$ds^2=\frac{1}{M^2}\left[-m\left(dt+\frac{s}{m}d\phi\right)^2+\frac{hm+s^2}{m}d\phi^2+\frac{dx^2}{hm+s^2}+dy^2\right], \quad (1)$$

where h,m,s are functions of only x and M depends only on y. Expanding the term enclosed in brackets we find

$$ds^2=\frac{1}{M^2}\left[-mdt^2-2sdtd\phi+hd\phi^2+\frac{dx^2}{hm+s^2}+dy^2\right], \quad (2)$$

which shows that the only condition for the metric to be nonsingular is

$$hm+s^2>0, \quad M\neq 0.$$

Thus, the function m can vanish despite the appearances in Eq. (1). Note that the metric would also have the correct Lorentzian signature when $hm+s^2<0$ with $h>0$ and $m<0$. However, in this case x would be a time coordinate and therefore these solutions would not be stationary, which is the case we are interested in.

From Eq. (1) it is obvious that there still remains some freedom in changing the coordinates but keeping the line element invariant. This allowed change is an arbitrary (nonsingular) linear change involving the coordinates t and ϕ, that is to say,

$$t=a_1\tilde{t}+a_2\tilde{\phi}, \quad \phi=b_1\tilde{t}+b_2\tilde{\phi}, \quad a_1b_2-a_2b_1\neq 0. \quad (3)$$

Performing this change and reading the new coefficients

$$g_{\tilde{t}\tilde{t}}\equiv -\frac{\tilde{m}}{M^2}, \quad g_{\tilde{t}\tilde{\phi}}\equiv -\frac{\tilde{s}}{M^2}, \quad \text{and } g_{\tilde{\phi}\tilde{\phi}}\equiv \frac{\tilde{h}}{M^2},$$

we have

$$\begin{pmatrix}\tilde{m}\\ \tilde{h}\\ \tilde{s}\end{pmatrix}=\begin{pmatrix}a_1^2 & -b_1^2 & 2a_1b_1\\ -a_2^2 & b_2^2 & -2a_2b_2\\ a_1a_2 & -b_1b_2 & a_1b_2+a_2b_1\end{pmatrix}\begin{pmatrix}m\\ h\\ s\end{pmatrix}. \quad (4)$$

From this we can evaluate $\tilde{h}\tilde{m}+\tilde{s}^2=(a_1b_2-a_2b_1)^2(hm+s^2)$ so that we must still perform the linear change in the coordinate x given by $\tilde{x}=|a_1b_2-a_2b_1|x$ in order to maintain *exactly* the same form (1).

Except for the particular cases studied below, the isometry group of the metric (1) is Abelian, two dimensional, and acts transitively on timelike two-surfaces. The two commuting Killing vector fields are

$$\frac{\partial}{\partial t}, \quad \frac{\partial}{\partial \phi}.$$

There is also a proper conformal Killing vector field given by $\partial/\partial y$, which also commutes with the two Killing vectors. This spacetime will be axially symmetric when there exists a Killing field which vanishes as a vector on a two-surface (the axis of symmetry) and satisfies the regularity condition (see, e.g., [5]). In order to find such a Killing vector, let us consider the change of variables (3) with $a_1=1$, $b_1=0$, which implies

$$\vec{\eta}\equiv\frac{\partial}{\partial\tilde{\phi}}=b_2\frac{\partial}{\partial\phi}+a_2\frac{\partial}{\partial t},$$

so that we are considering a general Killing vector field of the spacetime. In this case the transformation of the metric coefficients given in Eq. (4) takes the form

$$\tilde{h}=b_2^2h-a_2^2m-2b_2a_2s, \quad \tilde{s}=b_2s+a_2m, \quad \tilde{m}=m,$$

and the change in the variable x is written simply as $\tilde{x}=b_2 x$. In order to impose that $\vec{\eta}$ vanishes as a vector on a two-surface we demand that its scalar product with an arbitrary vector vanish on the axis of symmetry. In our case, this is clearly equivalent to the existence of a value x_1 of x where

$$\left.\begin{array}{l}\tilde{h}(x_1)=0\\ \\ \tilde{s}(x_1)=0\end{array}\right\}\Leftrightarrow\left\{\begin{array}{l}h(x_1)=-\left(\dfrac{a_2}{b_2}\right)^2 m(x_1),\\ \\ s(x_1)=-\dfrac{a_2}{b_2}m(x_1).\end{array}\right.$$

First of all, we note that $m(x_1)$ must be positive because we know that $\partial_{\tilde{t}}$ must be a timelike Killing vector at least in a neighborhood of the axis (notice that on points *off* the axis the two-surface of transitivity can be timelike with both $\partial_{\tilde{\phi}}$ and $\partial_{\tilde{t}}$ spacelike, but this is impossible *on* the axis where the transitivity surface becomes a one-dimensional line). Thus, the above two relations are in turn equivalent to

$$\frac{a_2}{b_2}=-\frac{s(x_1)}{m(x_1)},\quad (hm+s^2)(x_1)=0. \tag{5}$$

It is convenient to use the first relation in Eq. (5) to rewrite the function $\tilde{h}$ as

$$\tilde{h}=b_2^2\left(h+\frac{s^2}{m}+G(x)\right),\quad \text{where}\ \ G(x)=-\frac{1}{b_2^2}\frac{\tilde{s}^2}{m}.$$

In order to assure that the symmetry we are considering is indeed axial we must impose the regularity condition (also called elementary flatness condition), which demands

$$\frac{\nabla_\rho(\vec{\eta}^2)\nabla^\rho(\vec{\eta}^2)}{4\vec{\eta}^2}\to 1,$$

on the axis of symmetry. In our case, this condition takes the form

$$\lim_{\tilde{x}\to b_2 x_1}\frac{\tilde{h}\tilde{m}+\tilde{s}^2}{4\tilde{h}}\left(\frac{d\tilde{h}}{d\tilde{x}}\right)^2=1,$$

which can be rewritten in terms of the original functions and coordinates as

$$\frac{b_2^2}{4}m(x_1)\lim_{x\to x_1}\frac{h+\dfrac{s^2}{m}}{h+\dfrac{s^2}{m}+G}\left[\frac{d}{dx}\left(h+\frac{s^2}{m}+G\right)\right]^2=1.$$

Obviously, the necessary and sufficient condition for the symmetry to be axial is that the limit in this expression is finite and strictly positive because then we can fix the constant b_2 in order to satisfy the regularity condition. We already know that the function $h+s^2/m$ vanishes at $x=x_1$, and also that both the function G and its first derivative vanish there (remember that $\tilde{s}$ vanishes at $x=x_1$). Then, a careful analysis of the above limit near $x=x_1$ shows the following fundamental result.

Lemma 1. The metric (1) possesses a regular axis of symmetry at $x=x_1$ if and only if $m(x_1)>0$, $hm+s^2$ vanishes at $x=x_1$ and its first derivative is finite and nonzero there. In addition, the axial Killing vector is given by

$$\frac{2}{\sqrt{m(x_1)}}\frac{1}{\left|\dfrac{d}{dx}\left(h+\dfrac{s^2}{m}\right)\right|(x_1)}\left(\frac{\partial}{\partial\phi}-\frac{s(x_1)}{m(x_1)}\frac{\partial}{\partial t}\right).$$

It might seem that the vanishing of $hm+s^2$ at x_1 implies a singularity in the metric due to the term in dx^2 in Eq. (2). However, as is obvious from the intrinsic analysis we have just performed, that is not the case at all. The coordinate singularity in the metric can be solved trivially by making the change of coordinate $x\to x_1+X^2$, so that the line element becomes

$$ds^2=\frac{1}{M^2}\left[-m\,dt^2-2s\,dt\,d\phi+h\,d\phi^2+\frac{4X^2dX^2}{hm+s^2}+dy^2\right].$$

As is now obvious, the condition of lemma 1 is simply that both $hm+s^2$ and its first derivative vanish at $X=0$, and its second derivative is finite and nonzero there. Thus, the above line element is perfectly regular in g_{XX}. Nevertheless, we prefer to maintain the coordinate x because the line element takes the nice and more symmetric form (2) and also the field equations (see next section) are autonomous (they do not depend explicitly on the independent variable x).

Let us now extract some important consequences of lemma 1. When the metric coefficients satisfy the conditions of lemma 1 at two different values x_1 and x_2 and $hm+s^2$ is positive between these values, it follows that the spacetime has *two* different axes of symmetry. In this situation, the range of variation of the coordinate x must be restricted to $x_1\leqslant x\leqslant x_2$ and lemma 1 implies that the two vector fields

$$\frac{\partial}{\partial\phi}-\frac{s(x_1)}{m(x_1)}\frac{\partial}{\partial t},\quad \frac{\partial}{\partial\phi}-\frac{s(x_2)}{m(x_2)}\frac{\partial}{\partial t}$$

are proportional to axial Killing vectors (each one with a different axis of symmetry in principle) and, therefore, they have closed orbits. Unless they are proportional to each other, they generate a compact two-dimensional surface with Lorentzian metric which necessarily contains closed timelike curves (see [6]). Consequently, we must impose the condition

$$\frac{s(x_1)}{m(x_1)}=\frac{s(x_2)}{m(x_2)} \tag{6}$$

in order to have a physically well-behaved spacetime (i.e., satisfying causality conditions). In this case, there only exists one axial Killing vector field and its axis of symmetry has two connected components. In the coordinates t and ϕ adapted to the axial Killing vector, the condition that no closed timelike curves exist reads

$$h(x_1)=s(x_1)=0,\quad h(x_2)=s(x_2)=0.$$

III. EQUATIONS AND PROPERTIES OF THE PERFECT FLUID

Our aim now is to concentrate on those particular metrics (1) which have an energy-momentum tensor of a perfect fluid. As was first found in [2] and later in [3] the Einstein field equations in this case restrict the function $M(y)$ to satisfy

$$\left(\frac{dM}{dy}\right)^2=\epsilon a^2M^2-v,$$

where a and v are arbitrary constants and $\epsilon=\pm1$. This equation can be trivially solved to give one of the following possibilities depending on the sign of ϵ and the different values of the constants a and v [2]

$$\epsilon=+1,\begin{cases} M(y)=A\cosh(ay), & v=a^2A^2,\\ M(y)=Ae^{ay}, & v=0,\\ M(y)=A\sinh(ay), & v=-a^2A^2,\end{cases}$$

$$\epsilon=-1,\quad M(y)=A\cos(ay),\quad v=-a^2A^2,$$

$$a=0,\quad M(y)=Ay,\quad v=-A^2,$$

where A is an arbitrary nonvanishing constant and we have used the linear change of variables $y+$ const$\rightarrow y$ in order to avoid superfluous constants in $M(y)$. Analogously, the three functions $m(x)$, $h(x)$, and $s(x)$ must satisfy the two ordinary differential equations [2, 3]

$$h''m''+s''^2=0, \tag{7}$$

$$(hm+s^2)''+4\epsilon a^2=h'm'+s'^2, \tag{8}$$

where the prime means derivative with respect to x. Therefore, the perfect-fluid solutions of the form (1) depend on an *arbitrary* function of x.

The energy density and pressure are given by ($8\pi G=c=1$)

$$\mu=\tfrac{1}{4}M^2(h'm'+s'^2+4\epsilon a^2)+3v,$$

$$p=\tfrac{1}{4}M^2(h'm'+s'^2+4\epsilon a^2)-3v,$$

so that the perfect fluid satisfies the linear equation of state

$$\mu=p+6v. \tag{9}$$

The fluid velocity one-form can be written after some calculations as

$$\mathbf{u}=\frac{1}{M\sqrt{h'm'+s'^2+4\epsilon a^2}}[(-m\sqrt{-h''}+\epsilon_1 s\sqrt{m''})\mathbf{dt} -(\epsilon_1 h\sqrt{m''}+s\sqrt{-h''})\mathbf{d\phi}],$$

where $\epsilon_1\equiv \mathrm{sgn}(s'')$. From this it follows that the conditions

$$m''\geqslant0,\quad h''\leqslant0$$

must hold everywhere in order to describe true perfect fluids. In fact, from Eq. (7) we know that m'' and h'' must have opposite signs. Furthermore, the condition $\mu+p>0$ together with Eq. (8) avoid these functions to change simultaneously their sign. However, they do not impose the specific conditions $m''\geqslant0$ and $h''\leqslant0$. These come from the fact that one of the field equations is quadratic in the second derivatives and it includes some solutions which do not describe a perfect fluid (see [7] for a discussion). The expression for the fluid velocity vector can then be trivially evaluated and gives

$$\vec{u}=\frac{M}{\sqrt{h'm'+s'^2+4\epsilon a^2}}\left(\sqrt{-h''}\frac{\partial}{\partial t}-\epsilon_1\sqrt{m''}\frac{\partial}{\partial\phi}\right) =\frac{M\sqrt{-h''}}{\sqrt{h'm'+s'^2+4\epsilon a^2}}\left(\frac{\partial}{\partial t}+\frac{s''}{h''}\frac{\partial}{\partial\phi}\right), \tag{10}$$

where the second equality holds only at points with $h''\neq0$. Thus, the angular velocity Ω of the fluid is given by the expression $\Omega=s''/h''$, and the fluid is always nonconvective ($\vec{u}$ is on the $\{t,\phi\}$-planes), and differentially rotating except in the particular case $s''/h''=$const. These rigid rotation cases are thoroughly studied in Sec. V. Of course, in the rigid case the velocity vector of the fluid is shear free. In the general case, however, $\vec{u}$ is shearing and accelerating. The magnetic part of the Weyl tensor with respect to $\vec{u}$ vanishes [2]. Furthermore, the general solution is of Petrov type D with $\vec{u}$ lying in the two-planes generated by the two principal null directions of the Weyl tensor [2,3], so that the metrics belong to class I in Wainwright's classification of Petrov type D perfect fluids [10].

Let us now briefly make some remarks on the Einstein equations (7) and (8). This system of two differential equations for the three unknowns h, m, and s can be rewritten in an elegant way[1] by using the three-dimensional quadratic form

$$\begin{pmatrix} 0 & \frac{1}{2} & 0\\ \frac{1}{2} & 0 & 0\\ 0 & 0 & 1\end{pmatrix}, \tag{11}$$

which has signature $(-1,1,1)$. By defining the three-dimensional vector

$$\vec{v}(x)=(m,h,s),$$

Equations (7) and (8) can be rewritten, respectively, as

$$(\vec{v}''\cdot\vec{v}'')=0,\quad (\vec{v}\cdot\vec{v})''-(\vec{v}'\cdot\vec{v}')+4\epsilon a^2=0, \tag{12}$$

where the dot means scalar product using the metric (11). Therefore, the solutions of the perfect-fluid equations can be viewed as the trajectories of a point moving in a three-dimensional Minkowski spacetime with null acceleration vector, and the coordinate x is the parameter which describes the trajectory. It is clear that Eqs. (12) are invariant under the homothetic transformations of the three-dimensional Minkowski spacetime. This group of transformations in-

[1]We owe this elegant form to Dr. E. Ruiz.

cludes the three-dimensional Lorentz group and the dilations [which transform Eq. (11) into the same form except for a global positive constant factor ν^2]. Then, transforming the parameter x by $x \rightarrow \nu x$ the invariance of the system (12) is obvious. It might be thought that this invariance could give rise to new solutions given a particular one. This is not the case, though, because a direct calculation shows that the group of transformations given by Eq. (4), which are generated by the coordinate freedom (3), is exactly the homothetic group just discussed. Thus, the transformation of a solution of Eqs. (12) by the homothetic group gives exactly the same solution for the metric (1) written in another coordinate system. We will use this freedom to write some explicit solutions of (12) in its simplest possible form. The hitherto explicit solutions of the system (7), (8) are presented in Sec. V (for rigid rotation) and in the Appendix.

IV. ENERGY CONDITIONS AND THE LIMIT SURFACE

The study of stationary and axisymmetric perfect-fluid solutions has usually the main purpose of finding interior models which may describe the gravitational field inside a compact body (or at least a body with a limiting boundary which separates the self-gravitating fluid from its exterior). Given that the interior is intended to be a stationary body, the limit hypersurface S can be visualized as a two-surface at rest in the reference frame of the stationary Killing vector, which obviously implies that this hypersurface is timelike (or equivalently the normal vector is spacelike). Assuming that the exterior of the body is vacuum and the nonexistence of surface layers, the junction conditions between a perfect-fluid interior metric and its vacuum exterior imposes [8] that the pressure normal to the limit surface must vanish so that we find that the limit boundary is determined by the well-known condition of vanishing pressure:

$$p|_S = 0.$$

The interior of the body is then taken as the region where the pressure is positive (in order to describe the behavior of known fluids where the pressure tends to expand the body which is in equilibrium due to gravitational forces). Another physical condition on the perfect fluid is that the energy density is positive everywhere inside the body. Thus, in order to have a timelike hypersurface S of vanishing pressure with the density non-negative there, we must choose the constant v in the equation of state (9) such that $v \geqslant 0$, so that the dominant energy condition [6]

$$p \leqslant \mu, \quad \mu \geqslant 0$$

is fulfilled everywhere. Then, the only physically reasonable possibilities for $M(y)$ are

$$M(y) = A\cosh(ay), \quad v = a^2A^2, \quad p < \mu, \tag{13}$$

$$M(y) = Ae^{ay}, \quad v = 0, \quad p = \mu.$$

Our aim now is to show that the second possibility is a rather unusual case, so that we can restrict out attention to the first possibility. In order to see this, let us consider the case $v = 0$ and choose a point q located on S. The equation of state is $p = \mu$ everywhere and, given that the pressure is vanishing at q, we have that the density is also zero at this point. However, at any point where $\mu + p$ vanishes (and q is one of these points) the expression (10) for the fluid velocity $\vec{u}$ diverges and a detailed analysis of the energy-momentum tensor is necessary. It turns out that the momentum tensor is regular at q and takes the form

$$T_{\alpha\beta}|_q = k_\alpha k_\beta|_q ,$$

where the one-form $\mathbf{k}|_q$ reads, explicitly,

$$\mathbf{k}|_q = \frac{1}{\sqrt{2}}(-m\sqrt{-h''} + \epsilon_1 s\sqrt{m''})\mathbf{dt} - \frac{1}{\sqrt{2}}(\epsilon_1 h\sqrt{m''} + s\sqrt{-h''})\mathbf{d}\boldsymbol{\phi}|_q .$$

This one-form is null at q [obviously, assuming that the Einstein equations (7) and (8) hold]. In consequence, in the case $v = 0$ the energy-momentum tensor describes a fluid moving on timelike curves which tend to null curves as we approach the limit surface (defined as the points where the pressure vanishes), where the fluid is in fact moving at the speed of light with a nonvanishing energy density. This situation is not likely to occur in realistic bodies and the only way to avoid it is having a strictly vanishing Einstein tensor on the limit hypersurface. In this case, the body is more and more rarified and it transforms into the vacuum in a smooth way. This situation can only be accomplished by demanding that the null vector $\vec{k}$ is identically vanishing, which is clearly equivalent to

$$m''|_S = 0, \quad h''|_S = 0, \quad s''|_S = 0,$$

giving very particular solutions with a physically plausible interpretation.

Therefore, we will concentrate on the more general solution given in Eq. (13). It can be easily seen that the A can be set equal to 1 by redefining h, m, s, x, and a, so that $M = \cosh(ay)$. Now, it is convenient to define a new coordinate $Y(y)$ by means of

$$\sin(aY) = \tanh(ay),$$

which allows to rewrite the line element (2) in the form

$$ds^2 = dY^2 + \cos^2(aY)\left[-m dt^2 - 2s dt d\phi + h d\phi^2 + \frac{dx^2}{hm + s^2}\right]. \tag{14}$$

This form of writing the metric will be convenient later to interpret the coordinate Y (and thereby y). In terms of this coordinate, the density and pressure read

$$\mu = \frac{h'm' + s'^2 + 4a^2}{4\cos^2(aY)} + 3a^2, \quad p = \frac{h'm' + s'^2 + 4a^2}{4\cos^2(aY)} - 3a^2. \tag{15}$$

These expressions show that the spacetime (14) has a curvature singularity at

$$Y=\pm\frac{\pi}{2a}. \tag{16}$$

The equation determining the limit hypersurface of the fluid is given by

$$S:\frac{h'm'+s'^2+4a^2}{12a^2}=\cos^2(aY), \tag{17}$$

where the pressure vanishes. The range for the coordinate x where this equation has solution is clearly defined by $-4a^2<h'm'+s'^2\leqslant 8a^2$. Thus, the interior region of the self-gravitating body is given by

$$\frac{\pi}{2a}>|Y|\geqslant\frac{1}{a}\arccos\left(\sqrt{\frac{h'm'+s'^2+4a^2}{12a^2}}\right)\quad\text{when}\quad -4a^2<h'm'+s'^2\leqslant 8a^2,$$

$$-\frac{\pi}{2a}<Y<\frac{\pi}{2a}\quad\text{when}\quad h'm'+s'^2>8a^2. \tag{18}$$

We learn from these expressions that the curvature singularities (16) are always reachable going through the interior of the body. In the following sections we shall give a geometrical interpretation of the perfect-fluid metrics (14) discussing, in particular, the shape of the limit surface S and the location of the two singularities.

V. RIGIDLY ROTATING CASES

Let us now find the rigidly rotating solutions contained in the system (12). This is important because of two reasons. On the one hand, they will give the rigid limit of the differentially rotating solutions and, in particular, the static limit of the family, which in some sense will tell us what is the resulting spacetime when the rotation is ''switched off.'' On the other hand, these rigid solutions include the perfect-fluid metrics contained in Eq. (1) with a larger isometry group acting on the hypersurfaces $y=$ const. This follows because the angular velocity Ω of the perfect fluid is invariant under the action of the isometry group and, as Ω depends at most on the coordinate x, it must be constant (thus giving a rigid solution) when the isometry group acts on the hypersurfaces generated by the coordinates t, ϕ, and x. These more symmetric solutions may give us some insight into the interpretation of the general solution.

The rigidly rotating solutions are characterized by the condition Ω=const, and then we can always choose t such that the fluid velocity vector is proportional to ∂_t. From the explicit form of Ω and Eq. (7) it follows that

$$\text{rigid rotation:}\ \ s''=m''=0.$$

We must now distinguish two possibilities depending on whether m is a constant or depends explicitly on x. In the second case, we can perform a linear change on the variable x and use the freedom (3) with $b_1=0$ (in order to maintain $\vec{u}\propto\partial_t$) to set $m=x$ and $s=$const. Equation (8) can now be trivially integrated and the final solution is

$$h=C\ln\left(\frac{x}{b}\right)-4a^2x,\quad m=x,\quad s=\text{ const},$$

where $C\geqslant 0$ and $b>0$. It can be seen that the isometry group of this solution is two dimensional in general, and its energy density and pressure read

$$\mu=\frac{C}{4x\cos^2(aY)}+3a^2,\qquad p=\frac{C}{4x\cos^2(aY)}-3a^2,$$

so that $x=0$ is a singularity of the spacetime whenever it belongs to the allowed range for x. This metric was first found by one of us [9] as the most general stationary and axisymmetric rigidly rotating perfect fluid with Petrov type D and the fluid velocity vector lying in the two-planes generated at each point by the two principal null directions of the Weyl tensor (thus belonging to class I in Wainwright's classification [10]). The magnetic part of the Weyl tensor in the direction of $\vec{u}$ is zero and the existence of this solution was first proven by Collins [11], which in particular implied that a theorem due to Glass [12] was untrue. The static limit of this solution is obtained when $s=0$ and was first presented by Barnes [13]. This static limit is, however, not spherically symmetric.

There is also a special case with a larger group of symmetries defined by $C=0$ (for $s\neq 0$ since when $C=0$ and $s=0$ we have $hm+s^2\leqslant 0$ everywhere and the metric has not the signature we have been assuming throughout). This special metric is of positive constant curvature so that in fact it is locally the well-known de Sitter solution (although in a rather strange coordinate system). For the sake of completeness, we give explicitly the coordinate change from the coordinates we are using to more standard ones. It reads

$$T=\sqrt{\frac{s}{2a}}(t+2a\phi),\qquad \varphi=\sqrt{\frac{sa}{2}}(2a\phi-t),$$

$$\cos\chi=\sqrt{\frac{s+2ax}{2s}}\cos(aY),\qquad \sin\chi\sin\theta=\sqrt{\frac{s-2ax}{2s}}\cos(aY)$$

and brings the metric into the standard form

$$ds^2=-\cos^2\chi dT^2+\frac{1}{a^2}[d\chi^2+\sin^2\chi(d\theta^2+\sin^2\theta d\varphi^2)].$$

Obviously, we only obtain the portion of the de Sitter metric which is static since the metric in the original coordinates was static.

The analysis of the function $hm+s^2$ for this rigid solution shows that, in general $(C\neq 0)$, there appear two possible ranges for the coordinate x. One of these ranges always includes $x=0$, which is a curvature singularity as we have seen, and finishes at the first zero of $hm+s^2$. The other range appears only when

$$s^2<\frac{C^2}{16a^2},\quad \frac{\exp(\sqrt{1-16a^2s^2/C^2})}{1+\sqrt{1-16a^2s^2/C^2}}<\frac{C}{8ba^2}\leqslant\frac{\exp(-\sqrt{1-16a^2s^2/C^2})}{1-\sqrt{1-16a^2s^2/C^2}},$$

and then $hm+s^2$ vanishes at two other values of x being positive between them. Both these axes are regular, but the application of the remark following lemma 1 proves that they have different axial Killing vectors and therefore this spacetime (unless in the static case $s=0$) always contains closed timelike curves.

Let us now consider the other rigid case, when m is a constant. We can use the same coordinate freedom as before to set $m=1$ and $s=cx$, where c is an arbitrary constant (the other case of vanishing m does not give a perfect fluid). Equation (7) can be integrated and the full solution is

$$h=-\frac{c^2+4a^2}{2}x^2+\alpha x+\beta, \quad m=1, \quad s=cx,$$

where α and β are arbitrary constants. The expressions for the energy density and pressure of this solution are

$$\mu=\frac{c^2+4a^2}{4\cos^2(aY)}+3a^2, \quad p=\frac{c^2+4a^2}{4\cos^2(aY)}-3a^2,$$

which do not depend on the coordinate x. An easy calculation shows that this metric contains a four-dimensional isometry group and, therefore, it is worth saying some words about its geometry.

First of all, we note that the coordinates t and ϕ are still not uniquely determined. The allowed linear change (3) leaving h, m, and s invariant is

$$\phi\to A_1\phi, \quad t\to t+A_2\phi, \tag{19}$$

and the axis of symmetry of this solution is placed at the roots of the equation

$$hm+s^2=0 \Leftrightarrow \frac{c^2-4a^2}{2}x^2+\alpha x+\beta=0,$$

and these zeros must be simple in order to satisfy the regularity condition on the axis. Thus, the constants a, c, α, and β are restricted to satisfy the condition

$$\alpha^2-2\beta(c^2-4a^2)>0.$$

Depending on the sign of c^2-4a^2 (which can also vanish), there appear three different possibilities which, after using a change of coordinates of the type (19), a change of variable in the coordinate x, and some redefinitions of constants, can be written as

$$ds^2=dY^2+\cos^2(aY)\left\{-\left[dt+\tilde{c}\,\Sigma^2\left(\frac{r}{2},\kappa\right)d\phi\right]^2 +b^2\Sigma^2(r,\kappa)d\phi^2+b^2dr^2\right\}, \tag{20}$$

where b and $\tilde{c}$ are arbitrary constants, κ can take the values ± 1 or 0, we have put $a^2\equiv(\tilde{c}^2+8\kappa b^2)/16b^4$, and the function $\Sigma(r,\kappa)$ has the usual definition

$$\Sigma(r,\kappa)=\begin{cases}\sinh r & \text{if} \quad \kappa=-1,\\ r & \text{if} \quad \kappa=0,\\ \sin r & \text{if} \quad \kappa=1.\end{cases}$$

The metric (20) has a regular axis of symmetry at $r=0$ and its isometry group is four dimensional, acting multiply transitively on timelike hypersurfaces, and, therefore, the isotropy group at each point is a spatial rotation. The infinitesimal generators of the isometry group are given by

$$\vec{\xi}_1=\frac{\partial}{\partial\phi}\ (\text{axial Killing}), \quad \vec{\xi}_2=\frac{\partial}{\partial t},$$

$$\vec{\xi}_3=\sin\phi\frac{\partial}{\partial r}+\cos\phi\left(\frac{\Sigma'(r,\kappa)}{\Sigma(r,\kappa)}\frac{\partial}{\partial\phi}+\tilde{c}\frac{\Sigma^2(r/2,\kappa)}{\Sigma(r,\kappa)}\frac{\partial}{\partial t}\right),$$

$$\vec{\xi}_4=\cos\phi\frac{\partial}{\partial r}-\sin\phi\left(\frac{\Sigma'(r,\kappa)}{\Sigma(r,\kappa)}\frac{\partial}{\partial\phi}+\tilde{c}\frac{\Sigma^2(r/2,\kappa)}{\Sigma(r,\kappa)}\frac{\partial}{\partial t}\right),$$

where the prime in these expressions denotes derivative with respect to the variable r. The Lie algebra of this isometry group can be easily evaluated to give

$$[\vec{\xi}_1,\vec{\xi}_2]=[\vec{\xi}_1,\vec{\xi}_3]=[\vec{\xi}_1,\vec{\xi}_4]=\vec{0}, \quad [\vec{\xi}_2,\vec{\xi}_3]=\vec{\xi}_4,$$

$$[\vec{\xi}_2,\vec{\xi}_4]=-\vec{\xi}_3, \quad [\vec{\xi}_3,\vec{\xi}_4]=-\kappa\vec{\xi}_2-\frac{\tilde{c}}{2}\vec{\xi}_1.$$

Let us now make some considerations concerning this solution that will also be true for the general case when the isometry group is only two dimensional. Although we know that this solution has a well-defined axis of symmetry at any point of the spacetime because of the high symmetry it contains, we need only to consider those axes of symmetry such that the corresponding axial Killing vector is a linear combination of ∂_ϕ and ∂_t because this is the situation in the general solution. These axes of symmetry are located at $hm+s^2=0$ or, equivalently, where $\Sigma(r,\kappa)=0$. In the two cases $\kappa=0$ and $\kappa=-1$ the only value of r such that this expression vanishes is $r=0$ and, in consequence, the coordinate r can take arbitrarily large values. The norm of the axial Killing vector is given by the function h, which reads now

$$h(r)=b^2\Sigma^2(r,\kappa)-\tilde{c}^2\Sigma^4\left(\frac{r}{2},\kappa\right).$$

Obviously, this function vanishes at the axis $r=0$ and is strictly positive in a neighborhood of this axis. However, in the cases, $\kappa=0$ and $\kappa=-1$ (then necessarily $\tilde{c}\neq 0$), for large enough values of r this function becomes negative and, in fact, tends to $-\infty$ when $r\to+\infty$. Thus, in these cases, $\kappa=0$ or $\kappa=-1$, the solution contains closed timelike curves and violates causality. In the remaining case $\kappa=+1$ we have

$$(hm+s^2)(r)=b^2\Sigma^2(r,+1)=b^2\sin^2 r.$$

As this function vanishes again for $r=\pi$, the solution will have another axis of symmetry there. The axial Killing vector with the axis of symmetry at $r=\pi$ is given by the following combination of ∂_ϕ and ∂_t:

$$\frac{\partial}{\partial\phi}-\tilde{c}\frac{\partial}{\partial t}.$$

If $\tilde{c}\neq 0$, it is clear that the solution contains closed timelike curves (see the remark after lemma 1) so that the solution is not physically well defined. If $\tilde{c}=0$, this axial Killing vector coincides with the one with axis at $r=0$ [this corresponds to the situation when the condition (6) is satisfied] and the metric is perfectly well behaved. This particular metric with $\tilde{c}=0$ and $\kappa=1$ reads

$$ds^2=dY^2+\cos^2\left(\frac{Y}{\sqrt{2}b}\right)(-dt^2+b^2\sin^2 r\,d\phi^2+b^2dr^2).$$

The spacelike hypersurfaces $t=$ const are three-spheres. Thus, we can identify the geometrical meaning of the spacelike coordinates ϕ, r, and Y. In fact ϕ is clearly the azimuth of the three-sphere, r is the colatitude angle of each two-sphere $Y=$ const [these two-spheres have a radius given by $b^2\cos^2(Y/\sqrt{2}b)$] and Y is the latitude angle along any circumference $\phi=$ const, $r=$ const. The two connected components at $r=0$ and $r=\pi$ of the axis of symmetry of the axial Killing vector ∂_ϕ are part of a single curve because they meet at the north and south poles of the three-sphere, where $Y=\pm b\pi/\sqrt{2}$, respectively. This geometrical meaning of the spacelike coordinates will be of some help in the general cases studied in the next sections.

VI. ANALYSIS OF THE DIFFERENT TYPES OF SOLUTIONS

First of all, let us note that the geometrical meaning of ϕ is obviously standard and will hold for every spacetime described by the metric (14) with a regular axis of symmetry. The same happens for the interpretation of t because all the metrics are stationary. The interpretation of the coordinates x and Y is certainly more involved. We are now going to see that there exist two different cases of physical interest. In one of them the interpretation of the coordinates x and Y is similar to that of the more symmetric cases analyzed in the previous section. In the other, more interesting case, the coordinates x and Y can be visualized as similar to the typical bipolar coordinates of the Euclidian plane [14]. In order to show this we have to consider some of the consequences of the system of differential equations (7), (8).

Choosing the coordinates t and ϕ in the metric (14) so that ∂_ϕ is the true axial Killing vector, we know that both metric coefficients h and s vanish on the axis of symmetry located at some value $x=x_1$, which we can set without loss of generality at $x_1=0$ by performing a change of variables $x\rightarrow x+$ const. Thus we have $h(0)=s(0)=0$. We will concentrate in studying those solutions for which the interior region of the body reaches the axis of symmetry $x=0$. The reason for this lies in the fact that trying to interpret the geometry of an interior solution which does not contain the axis of symmetry is a much more speculative and unclear task. In fact, we could not even assure whether the solution is indeed axially symmetric or not until the exterior solution were known. Thus, we will assume $h'm'+s'^2>-4a^2$ at least in a neighborhood of the axis $x=0$ [see the expression for the pressure in Eq. (15)]. There appear then two completely different situations depending on whether

$$\text{case A:}\quad h'm'+s'^2>-4a^2,\quad \forall x>0, \tag{21}$$

so that for every x there exists a range of Y describing the interior of the body, or

$$\text{case B:}\quad \exists\overline{x}>0,\quad (h'm'+s'^2)(\overline{x})=-4a^2. \tag{22}$$

Case A will be the subject of this section. From a physical point of view, case B is much more interesting, and will be considered in detail in the last section.

From now to the end of this section we assume that Eq. (21) holds. We know that the axial Killing vector must be spacelike at least in a neighborhood of the axis and, actually, the regularity condition imposes $h'(0)>0$ so that h grows to positive values when x becomes positive. Furthermore, h'' is nonpositive everywhere so that the function h tends to stop its growing and h' possibly tends to cross again to negative values. In order to find all the possible behaviors for the functions h, m, and s which give physically well-motivated solutions, let us first suppose that h' becomes negative somewhere. Then, from the fact that $h''\leqslant 0$, there must exist another value $x_2>0$ where the function h vanishes. As discussed above, the only way in which this kind of behavior gives a physical spacetime (i.e., not containing closed timelike curves) is by also having $s(x_2)=0$. In this case, there only exists one axial Killing vector with an axis of symmetry with two connected components, and the coordinate x takes values inside the finite interval $(0,x_2)$. In consequence, the interpretation of x as an angle [like in the more symmetric solution (20) with $\kappa=1$] holds. By analogy with the more symmetric case, we can finally interpret the hypersurfaces $t=$ const in Eq. (14) as topologically equivalent to three-spheres with two connected components of the axis of symmetry which, in fact, belong to the same curve and meet at the two singular points $Y=\pm\pi/2a$ (located, respectively, at the north and south poles of the three-sphere). In this case, these two singularities can be viewed as two ''points'' located on the axis of symmetry. When a limit surface S of the fluid exists, it follows from the variation ranges of the coordinate Y describing the interior of the body that the region with positive pressure is given by the whole three-sphere except for a patch which is symmetric around the equatorial plane (it is clear from our interpretation of Y that $Y=0$ is the equatorial plane of the solution). Finally, it is obvious from Eq. (17) defining S and the fact that the range of variation of x is bounded that this two-surface (at each instant of time) is compact whenever it exists.

Having found which are the physical solutions in the case that h' becomes negative somewhere, we can now consider the remaining situation in which $h'\geqslant 0$ everywhere. From the fact that $h''\leqslant 0$ everywhere it follows that h' has a finite limit when $x\rightarrow+\infty$:

$$\lim_{x\rightarrow+\infty} h'(x)=\gamma^2\geqslant 0\Rightarrow \lim_{x\rightarrow+\infty} h''(x)=0,$$

where γ is a (possibly zero) constant. We are now going to show that in this case the self-gravitating body is either nonfinite or contains a singularity at some value of x. In both cases, the spacetime is not suitable to describe an isolated rotating body with an asymptotically flat exterior and therefore we will not try to interpret it.

Let us first suppose that $hm+s^2$ vanishes at some point $x_2>0$. From the remark after lemma 1 and the fact that now the function h is strictly positive for $x>0$, we have that the spacetime contains necessarily closed timelike curves and the solution is unphysical unless $(hm+s^2)'(x_2)=0$. When this happens, we know also by lemma 1 that $x=x_2$ is not a regular axis of symmetry of the spacetime. Let us show that the rotating body is infinite in size. In fact, the length of a curve $t=$ const, $\phi=$ const, and $Y=$ const completely contained in the interior of the body (it can be trivially seen that such a curve always exists in the case under consideration) from the axis of symmetry $x=0$ to the value $x=x_2$ is proportional to the expression

$$\int_0^{x_2} \frac{dx}{\sqrt{hm+s^2}}, \tag{23}$$

which diverges at least logarithmically given that both $hm+s^2$ and its derivative vanish at $x=x_2$. Thus, in this case the metric does not represent in any way a limited object because the size of the body is not finite, and we will not consider this kind of solutions any further in this paper (although nonfinite sources can be used sometimes to model situations such as accretion disks or similar).

Another possible behavior for $hm+s^2$ is that there exists some finite value $x=x_2$ where this function diverges (and does not vanish for $0<x<x_2$). Obviously, the function must diverge to $+\infty$ and it is clear that all the derivatives of $hm+s^2$ also diverge to $+\infty$ when we approach $x=x_2$. In particular, from the unboundedness of the second derivative of this function and the Einstein equation (8) it follows that $h'm'+s'^2$ also diverges to $+\infty$. The expressions for the density and pressure, Eqs. (15), show that the spacetime contains a true singularity at $x=x_2$ which, moreover, is located at a finite distance from the axis of symmetry given that the length (23) is now finite. This singularity is much worse than those located at $Y=\pm\pi/2a$ because this is an extended singularity fully contained inside the body. Therefore, this unphysical solution must be also ruled out in the description of rotating finite bodies.

We can thus move to the last possible behavior for $hm+s^2$; namely, this function remains positive and finite for all values $x>0$. We are going to show that, again, either the body is nonfinite or contains a singularity at a finite distance. In order to prove this, let us assume first that the spacetime is regular. This implies, in particular, that the density and pressure remain bounded for all $x>0$. Expressions (15) show that this happens if and only if there exists a positive constant K such that

$$h'm'+s'^2+4a^2<K+8a^2, \quad \forall x>0.$$

From Eq. (8) we find that this condition is in turn equivalent to

$$(hm+s^2)''<K, \quad \forall x>0.$$

Then, by integrating twice and remembering that $(hm+s^2)(0)=0$ we find

$$(hm+s^2)(x)<\frac{K}{2}x^2+\delta x, \quad \forall x>0,$$

where δ is an arbitrary positive constant. Therefore, we have

$$\int_0^\infty \frac{dx}{\sqrt{hm+s^2}} > \int_0^\infty \frac{dx}{\sqrt{x[\delta+(K/2)x]}}.$$

Now, the integral on the right-hand side is obviously divergent and, therefore, the integral on the left-hand side is also divergent. But this last integral is proportional to the proper length along an x-coordinate curve from the axis to $x=\infty$, and thus *if* the spacetime is regular, then the body is not of finite size. The other possibility is that the spacetime is singular at some value of $x>0$, and in this case this singularity can be trivially seen to be at a finite distance from the axis (even in the extreme case of the singularity placed at $x=+\infty$), as we had claimed. Thus, we have exhausted the whole of case A given in Eq. (21) and we can consider the more interesting case B.

VII. PHYSICALLY INTERESTING SOLUTIONS

In this section we analyze and interpret the solutions for case B defined in Eq. (22) assuming also that both h and $hm+s^2$ remain positive in the interval $(0,\overline{x})$. These last conditions on h and $hm+s^2$ can be assumed because otherwise the considerations made in the previous section would still hold. Now, the situation with respect to the coordinate x is quite different than before, as its range of variation has to be restricted to the interval $(0,\overline{x})$ not because x is an angular coordinate but rather because the body has a limiting surface.

In order to interpret the coordinates x and Y and to ascertain the shape of the body in this case, we first remark that, at each possible value of $\hat{x}\in(0,\overline{x})$, two different possibilities appear: either $(h'm'+s'^2)(\hat{x})>8a^2$ or $(h'm'+s'^2)(\hat{x})\leqslant 8a^2$. In the first case, and as follows from Eq. (18), the coordinate Y varies from $-\pi/2a$ to $\pi/2a$, and in particular $Y=0$ (which is the equatorial plane, as we shall presently see) belongs to the interior of the body at this particular $x=\hat{x}$. For all these $\hat{x}$, the hypersurfaces $x=\hat{x}$ go through the

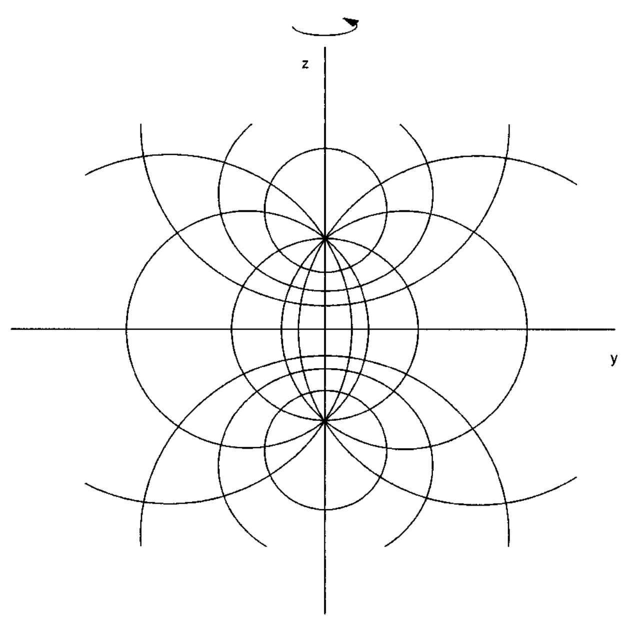


FIG. 1. System of bispherical coordinates for the Euclidian three-dimensional space.

interior of the body from the singularity at $Y=-\pi/2a$ up to the other singularity at $Y=\pi/2a$. In the second case, on the contrary, we see from Eq. (18) that the range of the coordinate Y has a symmetric ''hole,'' so to speak, around the value $Y=0$. This second possibility *always* appears in the case B under consideration as we approach the maximum value $\overline{x}$ of x [because of Eq. (22)]. For these values $\hat{x}$, the hypersurfaces $x=\hat{x}$ start at one singularity and go up to the limit surface S, then they cross the ''hole'' around $Y=0$ through the exterior of the body, then reemerge inside the body from another part of S, and finally reach the other singularity.

Thus, an interpretation of the shape and location of the two singularities $Y=\pm\pi/2a$ will give us a possible interpretation for x. Remember that, in the previously analyzed cases, these singularities were just single points placed symmetrically on the axis of symmetry. We are going to see that this interpretation still holds now. To that end, let us study the shape of the limit surface S *as seen from the exterior*. This can certainly be done if we assume that the metrics are matchable to some vacuum exterior at S, because then the matching conditions provide the exterior standard Weyl coordinates $\{\rho,z\}$ (see [15]) on the same surface S. This is a standard computation which produces, by using Eq. (17), the result

$$\rho(x)|_S=\frac{1}{12a^2}\sqrt{hm+s^2}(h'm'+s'^2+4a^2)|_S,$$

$$\frac{dz}{dx}\Big|_S=\pm\frac{2(8a^2-h'm'-s'^2)(h'm'+s'^2+4a^2)-3(h'm'+s'^2)'(hm+s^2)'}{24\sqrt{3}a^2\sqrt{8a^2-h'm'-s'^2}}\Big|_S,$$

where, of course, these formulas are valid only on the hypersurface S. As we can see from the first of these relations and Eq. (22), the Weyl cylindrical radius $\rho|_S$ vanishes at $\overline{x}$ (remember that $\rho=0$ is the axis as seen from the exterior). But this $\overline{x}$ corresponds to $Y=\pm\pi/2a$ on S, that is to say, to the singularities. Therefore, these two singularities are always two single points (at each instant of time) placed at the intersection of the limit surface of the body with the axis. The above formulas allow also for the explicit checking of all the assertions we shall make in what follows concerning the shape of the limit surface.

If we now remember that the hypersurfaces $x=\hat{x}$ go from one singularity to the other (with or without a ''hole''), we can visualize them, at each instant of time, as figures of revolution around the axis, all of them intersecting at the axis at the two points $Y=\pm\pi/2a$. In particular, the $(x=0)$-surface degenerates to a single line on the axis going from $Y=+\pi/2a$ down to $Y=-\pi/2a$. The surfaces $Y=$const at each time t are then the corresponding *orthogonal* family of surfaces of revolution around the axis, and thus, in particular, $Y=0$ corresponds to the equatorial plane and $Y=\pm\pi/2a$ degenerate to two single points. These coordinates $\{x,Y\}$ are therefore similar to the standard bipolar coordinates in the Euclidian plane [14], and the set $\{x,Y,\phi\}$ is analogous to the system of bispherical coordinates in flat three-space [14], as can be checked in Fig. 1 where we represent them.

Once we have given a precise interpretation for the coordinates $\{x,Y\}$ in case B, we can now study and classify the different shapes and properties of the interior body and its limit surface. Of course, given the enormous freedom still available in choosing the functions h,m, and s, the possibilities are endless. However, we can certainly provide a qualitative classification of the different cases of interest that may appear.

A. Typical isolated compact bodies with equatorial symmetry

This case arises when $(h'm'+s'^2)(0)>8a^2$ and there only exists one positive value $x=x_2<\overline{x}$ in which $(h'm'+s'^2)(x_2)=8a^2$ [the existence of at least one such value is necessary from Eq. (22)]. In this case, the shape of the limit surface is presented in Fig. 2: The axis from one singularity to the other is completely contained in the body, which presents manifestly equatorial symmetry [see Eq. (17)]. The proper length from the axis to the surface along the equatorial plane (the ''equatorial radius'') is given simply by

$$R_{\text{eq}}=\int_0^{x_2}\frac{dx}{\sqrt{hm+s^2}}, \tag{24}$$

while the proper distance from the center $x=Y=0$ to one of the singularities (the ''radius along the axis'') is

$$R_{\text{ax}}=\frac{\pi}{2a}. \tag{25}$$

The ''north and south poles'' are the singularities and the shape of the limit surface is oblatum, prolatum, or irregular depending on the explicit forms of the functions h,m, and s and on the constant a, as is obvious from Eqs. (24) and (25). Notice that, in general, the shape of the limit surface can adopt very irregular forms and, in fact, the distances orthogonal to the axis and up to the surface S can have several local maxima. Some examples illustrating these behaviors are shown in Fig. 3(a) and Fig. 3(b).

B. Isolated compact bodies with equatorial symmetry and a centered hole

This case arises when $(h'm'+s'^2)(0)<8a^2$ and there exist just a couple of values x_2 and x_3 $(0<x_2<x_3<\overline{x})$ such that $h'm'+s'^2-8a^2$ changes its sign at $x=x_2$ and $x=x_3$. Notice that, in this case, these values must appear in pairs because of Eq. (22). The limit surface has now two connected components, the inner and the outer surfaces of the body as shown for a typical case in Fig. 4. The shapes of

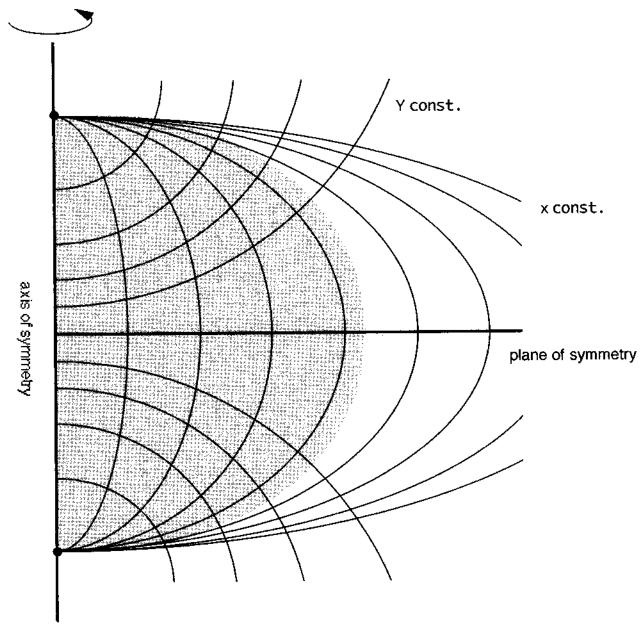
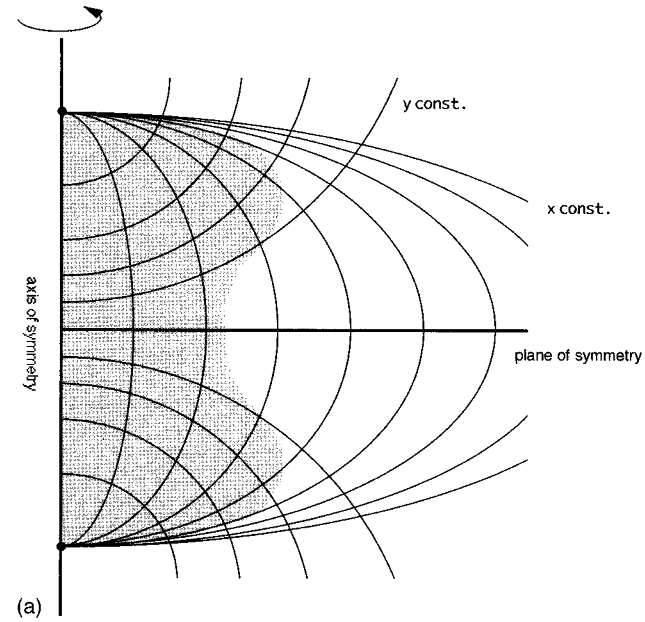


FIG. 2. Typical isolated body with equatorial symmetry. The shadowed region represents the interior of the body and the two singularities in the north and south poles are indicated by two dots. The coordinate lines x const and Y const, which are similar to the bispherical coordinates, are also shown. Notice that we have chosen these coordinates such that all the x const lines reach the singularities tangent to one another. This means that the coordinates $\{x,Y\}$ do not cover the whole two-dimensional plane. Any other similar interpretation for $\{x,Y\}$ could be possible with the only result of changing the form of the bodies as seen from the exterior but without changing the main features of the interior body. The vertical line through the two singularities is the axis of symmetry and the three-dimensional picture is obtained by rotating the figure around this axis, as indicated by the circular arrow. This will be the case in all the following figures.

these two connected components can vary as much as in the previous case, and both of them can adopt very regular or irregular forms. However, the inner surface is always completely regular (it has no singularities). Obviously, this type of configuration for a self-gravitating perfect fluid can be in equilibrium only because of the differential rotation. In this case, there are no such concepts as equatorial radius, etc., but some other typical appropriate distances can be computed without difficulty (such as the proper distance between the inner and outer surfaces along the equatorial plane).



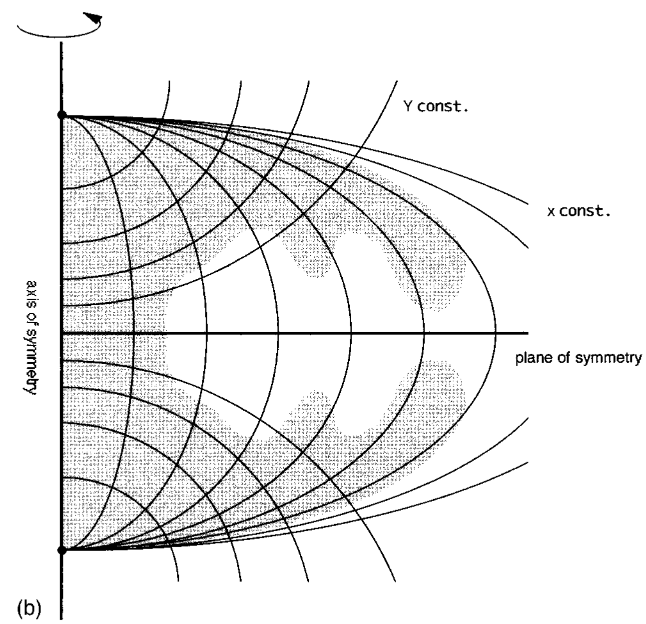


FIG. 3. (a) Isolated compact body with equatorial symmetry and a more irregular profile. The allowed shapes are those in which the coordinate lines x const entering inside the body when coming from the equatorial plane remain inside the body until they reach the singularities. (b) Another example with an even more irregular shape. This figure indicates how the cases with either centered holes or toroidal holes (see Figs. 4–7) may appear.

C. Isolated compact bodies with equatorial symmetry and a toroidal hole

This case is defined by $(h'm'+s'^2)(0)>8a^2$ together with the existence of exactly three values x_2,x_3, and x_4 $(0<x_2<x_3<x_4<\overline{x})$ where $h'm'+s'^2-8a^2$ changes its sign. Now, the axis from one singularity to the other is again completely contained in the body. However, there is a hole around the equatorial plane for all $x\in(x_2,x_3)$. Because of the axial symmetry, this hole appears as similar (topologically) to a standard torus. A typical example is represented in Fig. 5, where we can see that there are again two connected components of the limit surface; the inner one does not intersect the axis and is completely regular. The exact forms of the inner and outer surfaces depend completely on the specific functions h,m, and s, and there is a great variety of possibilities.

The distance along the axis from the ''center'' $x=Y=0$ up to the ''north pole'' singularity is exactly Eq. (25) again. On the other hand, the equatorial distance from the axis to the outer surface cannot be computed now because we need to know the exterior metric in the toroidal hole. Thus, the outer appearance of this body (prolatum, oblatum, etc.) seems to be uncertain at this stage. Nevertheless, we can

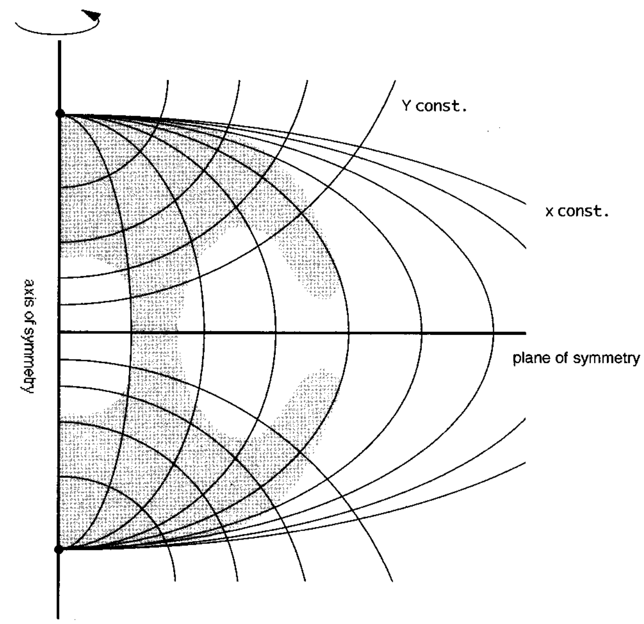


FIG. 4. Isolated compact body with equatorial symmetry and a centered hole. Now the axis is not completely contained inside the body.

always give a lower bound for the equatorial radius defined by the sum of the equatorial distance from the axis to the inner surface plus the distance from this inner surface up to the outer surface, that is to say,

$$R_{\mathrm{eq}} > \int_0^{x_2} \frac{dx}{\sqrt{hm+s^2}} + \int_{x_3}^{x_4} \frac{dx}{\sqrt{hm+s^2}}. \tag{26}$$

This lower limit for R_{eq} together with its evident generaliza-

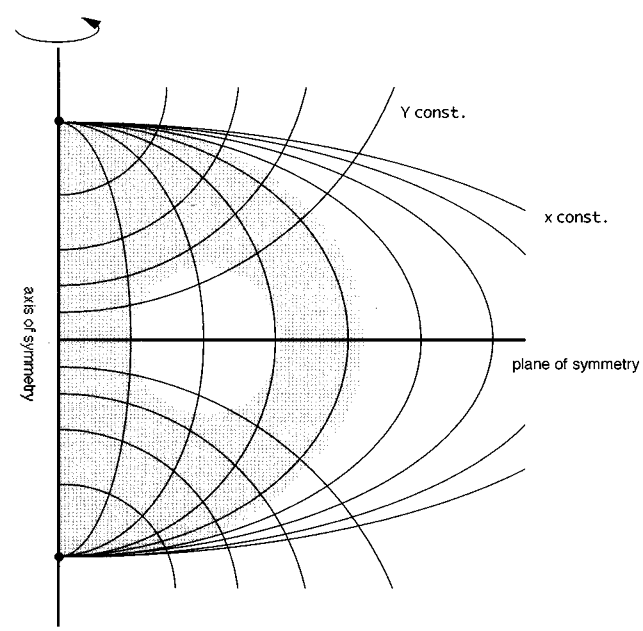


FIG. 5. Isolated compact body with equatorial symmetry and a toroidal hole.

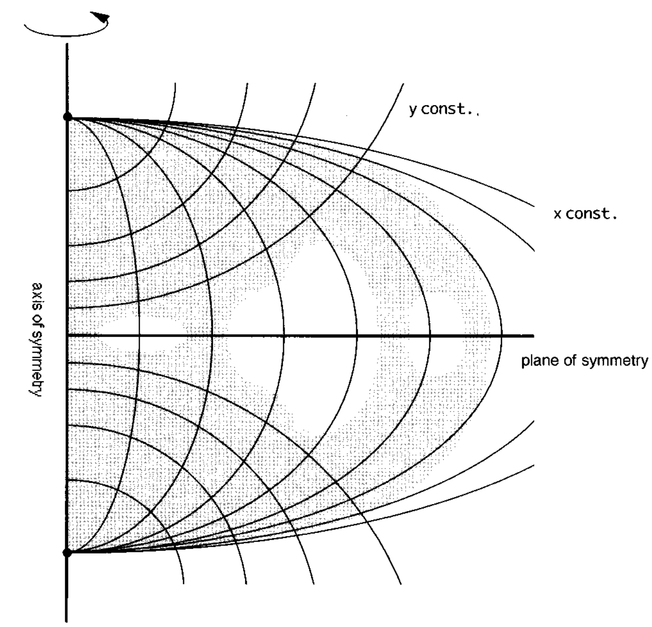


FIG. 6. Isolated compact body with equatorial symmetry and three toroidal holes. Notice that an arbitrary number of interior toroidal holes are allowed (see main text). These toroidal holes can also have many different shapes.

tion to other distances orthogonal to the axis allows to ascertain the outer form of the body in some situations.

D. Compact bodies with equatorial symmetry and a set of toroidal holes

This case is the generalization of the previous one to the existence of a finite number (say, n) of toroidal holes in the interior of the body. Thus, it is defined by the conditions $(h'm'+s'^2)(0) > 8a^2$ together with the existence of exactly $2n+1$ values

$$x_2, x_3, \dots, x_{2n+2}(0 < x_2 < x_3 < \cdots < x_{2n+2} < \overline{x})$$

where $h'm'+s'^2-8a^2$ changes its sign. The axis from one singularity to the other is completely contained in the body, and there appear n holes around the equatorial plane, each of them for the values $x \in (x_{2j}, x_{2j+1})$, with $j=1,2,\dots,n$. Because of the axial symmetry, each of these holes resembles (topologically) a standard torus. A simple example of this case is presented in Fig. 6, where we see that now there are $n+1$ connected components of the limit surface: n of them are inner surfaces and correspond to each of the toroidal holes and the remaining one is the outer surface. None of the n inner surfaces intersect the axis, and all of them are completely regular. The exact forms of the inner and outer surfaces depend again on the specific forms of h, m, and s.

Expression (25) gives again the distance along the axis from the "center" $x=Y=0$ up to the "north pole" singularity. Concerning the equatorial distance from the axis to the outer surface, we can only give a lower bound as in the previous case. This lower bound is the trivial generalization of that defined by Eq. (26) and reads

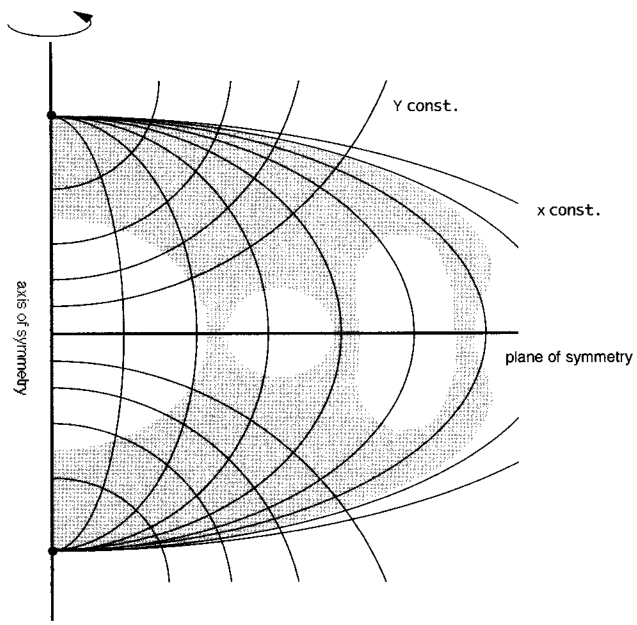


FIG. 7. Isolated compact body with equatorial symmetry, one centered hole, and two toroidal holes. Again an arbitrary number of toroidal holes is allowed and many different shapes for them are possible.

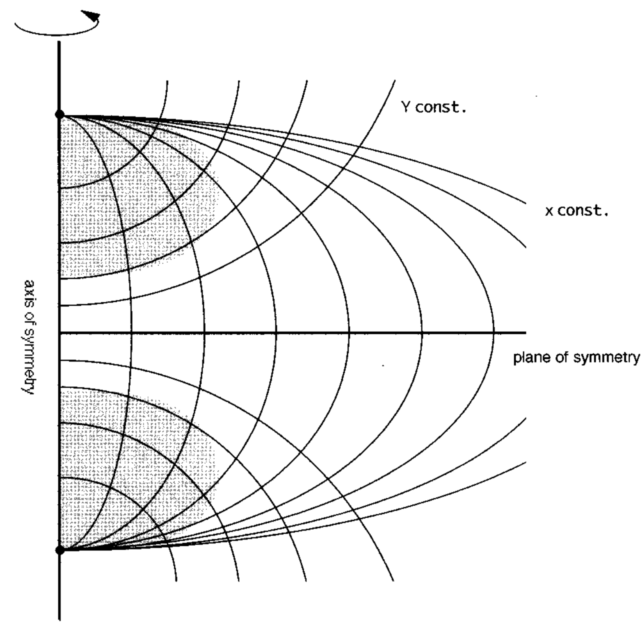


FIG. 8. Compact bodies with no equatorial symmetry. In this case no point on the equatorial plane belongs to the interior of the body and, therefore, the interior has two connected components each one containing one of the singularities.

$$R_{\rm eq} > \int_0^{x_2} \frac{dx}{\sqrt{hm+s^2}} + \sum_{j=1}^{n} \int_{x_{2j+1}}^{x_{2j+2}} \frac{dx}{\sqrt{hm+s^2}}.$$

The comparison of this lower limit for $R_{\rm eq}$ with $R_{\rm ax}$ of Eq. (25) gives information on the outer shapes of the body.

E. Compact bodies with equatorial symmetry and central and toroidal holes

In this case we have $(h'm'+s'^2)(0)<8a^2$, so that there is a central hole, and also we allow for the existence of n toroidal holes. Thus, there exist exactly $2(n+1)$ values $x_2, x_3, \cdots, x_{2n+3}$ $(0<x_2<x_3<\cdots<x_{2n+3}<\overline{x})$ where $h'm'+s'^2-8a^2$ changes its sign. The n holes around the equatorial plane are defined then by the values $x \in (x_{2j+1}, x_{2j+2})$, with $j=1,2,\ldots,n$. The central hole is obviously defined by the values $x \in (0, x_2)$. Now, there are $n+2$ connected components of the limit surface, $n+1$ of them are inner surfaces and the remaining one is the outer surface. Among the $n+1$ inner surfaces, n of them correspond to the toroidal holes and the other to the central hole. All the inner surfaces are completely regular. These properties can be seen in the example represented in Fig. 7.

F. Compact bodies with no equatorial symmetry

This is, in fact, the last qualitatively different possibility. It is uniquely defined by the condition $(h'm'+s'^2)(x) < 8a^2$ for all possible $x \in (0, \overline{x})$. Thus, the ''hole'' around the equatorial plane appears for all values of x, and therefore no single point of the equatorial plane belongs to the interior of the body. It follows that, actually, the body has two different connected parts, as ''two drops of water,'' placed symmetrically with respect to the equatorial plane, each of them containing one of the singularities $Y=\pm\pi/2a$; see Fig. 8. Nevertheless, we must choose just one of the connected interiors (both are identical upon reflection) as the interior of the body, because the representation of the exterior between ''drops'' cannot be guaranteed unless we know the vacuum exterior solution valid at that region. In order to fix ideas, we shall always choose the ''drop'' with positive values of Y. Thus, now the interior of the body is simply the interior of the upper ''drop,'' which does *not* have equatorial symmetry, as is manifest. These interiors have the advantage that they only contain one singularity, placed at the ''north pole,'' the rest of the limit surface being completely regular. Their exterior appearance can adopt many different shapes, as, for example, the irregular form given in the example of Fig. 9. In general, this family of solutions can be thought as the limit of the first family given in this section (i.e., typical isolated compact bodies with equatorial symmetry) when $x_2 \to 0$, so that the equatorial radius (24) for those metrics approaches zero. In the limit, there appear two bodies touching each other tangentially at a single point in the center $x=Y=0$. When this point disappears the new family we are considering arises.

Another interesting remark regarding this family without equatorial symmetry is that, by making $h'm'+s'^2$ closer and closer to $-4a^2$ for all x, we get smaller and smaller ''drops'' placed around the singularities $Y=\pm\pi/2a$. In the limit $(h'm'+s'^2)(x)=-4a^2$, we would simply have the two points $Y=\pm\pi/2a$ as the interior body. In fact, it can be seen that in this limit the metric is regular and corresponds to the de Sitter metric which appeared in the rigid cases studied in Sec. V.

Apart from the Wahlquist family of perfect-fluid solutions [18–20] (which have rigid rotation), and as far as we know, the models presented in this section are the best available explicit solutions describing the interior of axially symmetric

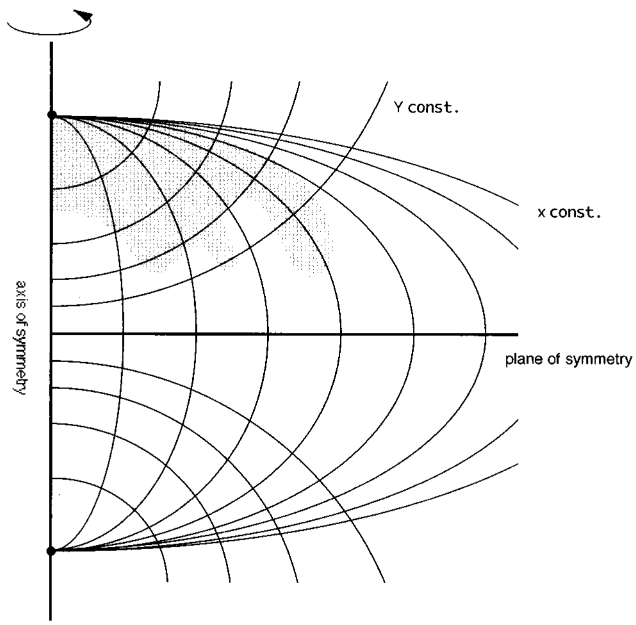


FIG. 9. Compact body with no equatorial symmetry and a very irregular shape. This body has only one singularity.

differentially rotating isolated compact bodies. It remains the question of the matching of these models to some appropriate exterior solutions.

ACKNOWLEDGMENTS

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APPENDIX: OTHER EXPLICIT SOLUTIONS

Before finishing this paper let us briefly discuss the possibility of finding explicit solutions of the system (12). Even though the system seems to be not very difficult to handle, it has a geometrical interpretation as trajectories in a three-dimensional Minkowski spacetime, and it contains an arbitrary function in its general solution, it is very difficult indeed to find explicit functions h, m, and s satisfying the system. Apart from the rigid metrics we have explicitly written above, only two other particular solutions of the system have been found. The first one was given in the paper where the family was presented [2] and reads

$$m = x x^{\sqrt{19/10}} (x^{1/\sqrt{10}} - \epsilon a^2 l x^{-1/\sqrt{10}}),$$

$$h = x x^{-\sqrt{19/10}} (r x^{1/\sqrt{10}} - \epsilon a^2 n x^{-1/\sqrt{10}}),$$

$$s = x (x^{1/\sqrt{10}} - \epsilon a^2 k x^{-1/\sqrt{10}}),$$

where the constants l, r, n, and k are given by the values

$$l = \tfrac{20}{2907}(19\sqrt{10} - 64)(23 + 5\sqrt{19}), \quad r = -\tfrac{1}{216}(19\sqrt{10} + 64),$$

$$n = \tfrac{5}{323}(23 - 5\sqrt{19}), \quad k = \tfrac{360}{323},$$

[in the original paper [2] there also appeared two arbitrary constants which, however, can be absorbed into the coordinates by using a transformation of the type (4)]. As explained in previous sections we can restrict ourselves to the case $\epsilon = +1$. For this solution there exist two disconnected regions where $hm + s^2 > 0$. They are given by

$$0 \leqslant x \leqslant \left(\frac{1080(5 - \sqrt{19}) a^2}{323(8 - \sqrt{10})} \right)^{\sqrt{10}/2}$$

or

$$\left(\frac{1080(5 + \sqrt{19}) a^2}{323(8 - \sqrt{10})} \right)^{\sqrt{10}/2} \leqslant x \leqslant +\infty.$$

It can be checked by direct calculation that in both regions the inequality

$$h' m' + s'^2 \geqslant -4a^2$$

holds. The solution in the first region has a regular axis of symmetry at the nonzero extremum of the interval and contains a singularity at $x = 0$ which is located at a finite distance from the axis of symmetry. Similarly, the solution in the second region has a regular axis of symmetry at the finite extremum of the interval and contains a singularity at $x = +\infty$ also located at a finite distance from the axis. Thus, as discussed above, none of these explicit solutions are physically reasonable.

The other explicitly known solution was found by García [16] in the particular case $a = 0$ (which consequently does not give a physically well-behaved solution, as discussed in the text). The metric coefficients are now

$$m = x^{-\nu+\beta}, \quad h = x^{\nu+\beta}, \quad s = x^{\beta} \sqrt{\frac{2(1-2\beta)^3}{\beta(3\beta-2)^2}},$$

where β is a constant satisfying

$$0 < \beta < \tfrac{1}{2}$$

and ν is a positive constant given by

$$\nu^2 = \beta^2 + \frac{2(\beta-1)^2(1-2\beta)}{2-3\beta}.$$

In the paper where this solution was presented there appeared two other arbitrary constants which again can be reabsorbed in the coordinates. Evaluating $hm + s^2$ for this solution we find

$$hm + s^2 = \left(1 + \frac{2(1-2\beta)^3}{\beta(3\beta-2)^2} \right) x^{2\beta},$$

so that the axis of symmetry would be located at $x = 0$ if it exists. The derivative of this function diverges at $x = 0$ and, therefore, because of lemma 1, we can say that the solution is not axially symmetric.

These solutions have a particularity which is worth comment. The explicit expressions for the density and pressure of these solutions show that they diverge when the coordinate x tends to zero. Thus, the would-be axis of symmetry is a true singularity of the spacetime and the regularity conditions we imposed above cannot be applied in this singular case (in the definition of axial symmetry it is assumed the existence of a two-surface of fixed points in the manifold so that they are necessarily regular points). We point out that in the case of singular axes of symmetry there does not exist an appropriate theory to decide when a given singular spacetime is indeed representing a singular axially symmetric spacetime or not (see [17] for a more detailed discussion).

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