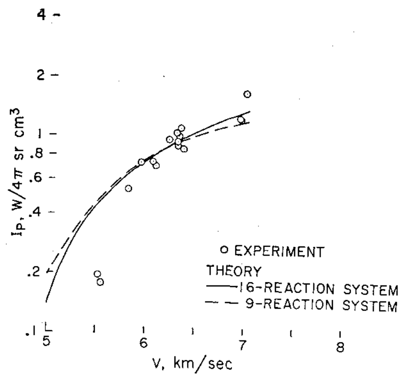

Fig. 2 Experimental and computed peak CN intensities for 25% CO_2 − 75% N_2; p_1 − 350μ Hg.

factor of 2 appear in the nonequilibrium region, thus illustrating the uncertainties regarding the behavior of species other than CN. Except for their equilibrium levels, the curves for N and NO are omitted from the figure for clarity, but their behavior in time and magnitude is similar to that of O and CN, respectively, and in good agreement with the 16-reaction system. It is also shown in Ref. 1 that both the experimental data and the computed results follow binary scaling laws in the nonequilibrium region. Hence, the effects of varying ambient density are accurately reproduced, and it is only necessary to compare results obtained with the two chemical models at other shock speeds and gas mixtures. For these comparisons, the magnitude and time of the peak intensity are indicative of the adequacy of the 9-reaction system, and if they agree with the data, the remainder of the intensity profile is also found to be matched. The variation of the peak CN intensity over the range of shock speeds considered was computed using the 9-reaction system of Table 1 and is compared with computations using the 16-reaction system and experimental data in Fig. 2. The results show the predictions made with the 9- and 16-reaction systems to be similar in the speed range from 5.8 to 7.0 km/sec, which is that of primary interest for Mars entry. The computed and experimental times to peak CN intensity are shown in Fig. 3. Again, the results for both reaction systems are similar and provide a good approximation to the experimental data.

The foregoing results indicate that the predictions based on the simplified chemical model of 9 reactions presented in Table 1 are adequate for estimating the nonequilibrium radiant emission of CN in shock-heated mixtures of CO_2 and N_2 for the range of speeds of current interest for entry into the Martian atmosphere.

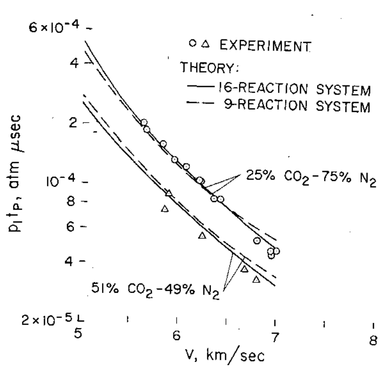

Fig. 3 Experimental and computed times to peak CN intensity.

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Some Remarks on the Dynamics of Deformable Bodies

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A RECENT paper by Professor Ashley,[1] which deals with the dynamic behavior of large, flexible bodies in orbit, raises a number of interesting points in connection with the general problem of the dynamics of deformable bodies. A deformable body in this context is taken to mean one for which the relative elastic displacements are so small that only first-order terms in relative displacement are retained in the analysis; the body is, however, allowed an over-all motion which is completely unrestricted.

This note takes issue on three questionable aspects of Professor Ashley's analysis, and although these are of general significance,[2,3] it will be convenient to frame the discussion by detailed reference to Professor Ashley's paper; to this end the notation will be, wherever possible, identical to that of Ref. 1.

The three issues concern firstly, the definition of the over-all linear and angular velocities of a deformable body; secondly, the free motion of a spinning, deformable body and; thirdly, the appropriate form of elemental equation of equilibrium to be used for spinning bodies. Let us deal with these in turn.

1. Principal and Mean Body Axes

It is implied in several sources[1,2,4] that the use of a superposition of free vibration modes to represent the displacement field of a deformable body means that the over-all motion of the body is described by the motion of a set of "principal axes" for the deforming body; this is not true except under special circumstances.

For a deformable body there is no obvious geometric way of defining the over-all linear and angular velocities $\mathbf{v}$ and $\mathbf{\Omega}$ of the body relative to inertial axes. This question of defining what exactly is meant by $\mathbf{v}$ and $\mathbf{\Omega}$ is dealt with in Ref. 5, where, out of all the possible choices of axis systems, three are considered in detail, namely attached, mean, and principal axes. The first of these shall not concern us here.

Let the instantaneous position vector of a material point from the origin of body axes be $\mathbf{r} = \bar{\mathbf{r}} + \mathbf{q}$ where $\bar{\mathbf{r}}$ is the position vector in a reference state and $\mathbf{q}(\bar{\mathbf{r}},t)$ is the relative displacement vector. The conditions for mean axes are that the deformation motion has zero linear and angular momentum relative to the body axes; that is,

$$\int_V \mathbf{q}\Delta m = \mathbf{0} \tag{1}$$

and

$$\int_V \bar{\mathbf{r}} \times \mathbf{q}\Delta m = \mathbf{0} \tag{2}$$

for all t. For principal axes, condition (1) is retained (stationary c of m relative to the body axes), but condition (2) is replaced by the requirement that the change in the inertia tensor due to the deformation should, at all times, be diagonal. Let $\tilde{\mathbf{\Psi}}$ be the reference inertia tensor and $\mathbf{\Psi}$ the total (in-

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stantaneous) inertia tensor relative to body axes, then $\boldsymbol{\Psi}' = \tilde{\boldsymbol{\Psi}}' + \boldsymbol{\Psi}'_q$ where

$$\boldsymbol{\Psi}'_q = \int_V [2\bar{\mathbf{r}}.\mathbf{q}\mathcal{I} - (\mathbf{q}\bar{\mathbf{r}} + \bar{\mathbf{r}}\mathbf{q})]\Delta m$$

For principal (deformation) axes it is required that $\boldsymbol{\Psi}'_q$ be diagonal; that is,

$$\mathbf{i}\cdot\boldsymbol{\Psi}'_q\cdot\mathbf{j} = \mathbf{j}\cdot\boldsymbol{\Psi}'_q\cdot\mathbf{k} = \mathbf{k}\cdot\boldsymbol{\Psi}'_q\cdot\mathbf{i} = 0 \tag{3}$$

for all t. If the gross location of the axis system in the reference configuration is chosen so that $\tilde{\boldsymbol{\Psi}}'$ is diagonal, then $\boldsymbol{\Psi}'$ will remain diagonal for all t.

Since there are no over-all kinematic boundary conditions on the body, it is important to note that a given strain field determines the corresponding displacement field only up to a small translation and rotation. In this sense, for any given displacement field $\mathbf{q}$, conditions (1) and (2) or (1) and (3) uniquely define the linear and angular velocities of the body.

In terms of the Cartesian components of $\bar{\mathbf{r}}$ and $\mathbf{q}$, conditions (2) and (3) are, respectively,[5]

$$\int_V(\bar{y}q_z - \bar{z}q_y)\Delta m = \int_V(\bar{z}q_x - \bar{x}q_z)\Delta m = \int_V(\bar{x}q_y - \bar{y}q_x)\Delta m = 0 \tag{2a}$$

and

$$\int_V(\bar{y}q_z + \bar{z}q_y)\Delta m = \int_V(\bar{z}q_x + \bar{x}q_z)\Delta m = \int_V(\bar{x}q_y + \bar{y}q_x)\Delta m = 0 \tag{3a}$$

It is pointed out in Ref. 5 that mean and principal axes may often coincide for lenticular or prismatic bodies in which transverse displacement is the main deformation mode. But that they do not in general coincide is readily illustrated by the following simple example.

Let the beam of Fig. 1a, which carries two end masses, bend in the antisymmetric (fixed) mode $(d^2q_y/d\bar{x}^2)\cdot F(t)$ where $F(t)$ is an arbitrary function of time. In this case application of Eqs. (2) and (3) shows that mean and principal axes coincide, and these axes are sketched in Fig. 1b. Now let the end masses have significant moment of inertia about their centroidal axes as sketched in Fig. 2a. Then in order that the nett angular momentum relative to mean axes should be zero, the whole beam must clearly be rotated counterclockwise relative to the attitude of Fig. 1b as shown in Fig. 2b [Eq. (2a)]. But in order that the nett product of inertia should be zero (principal axes), the whole beam must be rotated an equal amount in a clockwise direction as shown in Fig. 2c [Eq. (3a)]. If the given bending mode were a free vibration mode, then naturally the mean axes of Fig. 2b would be at rest, but the principal axes of Fig. 2c would exhibit a small oscillatory rotation equal in frequency to the natural frequency.

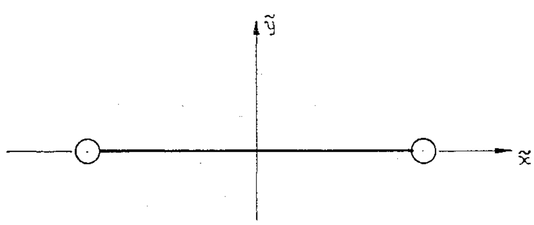

Fig. 1a Beam with end masses.

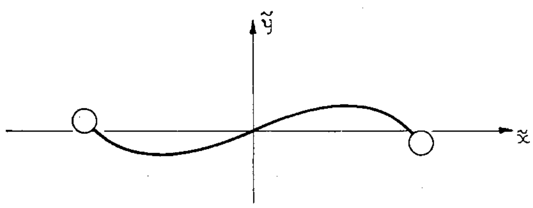

Fig. 1b Mean and principal axes coincide.

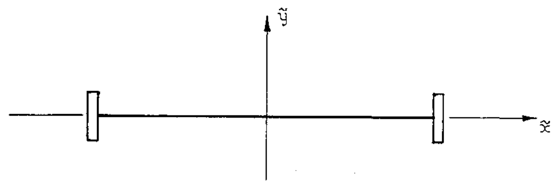

Fig. 2a Beam with end masses having significant moment of inertia.

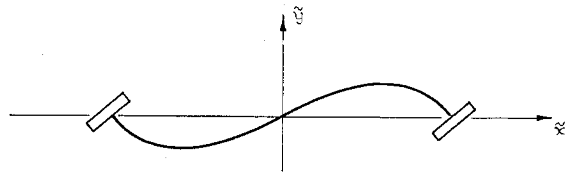

Fig. 2b Mean axes.

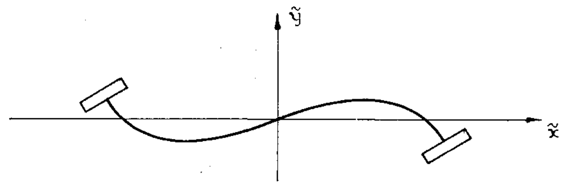

Fig. 2c Principal axes.

2. Free Motion of a Spinning Deformable Body

In Sec. 4 of Ref. 1, Professor Ashley, using equations of motion from Ref. 2 in which a number of terms are omitted "for simplicity," comes to the conclusion that, for a steadily spinning body, "the rotational equation supplies the well-known information that $\boldsymbol{\Omega}_o$ must be along a principal axis to ensure an equilibrium solution." The position is not as simple as this.

If free vibration modes are used to describe the displacement so that $\boldsymbol{\Omega}$ is defined, ipso facto, with respect to mean axes, then a steady rotation is possible only if

$$(\delta\boldsymbol{\Psi}'_q/\delta t)\cdot\boldsymbol{\Omega}_0 + \boldsymbol{\Omega}_0 \times [(\tilde{\boldsymbol{\Psi}}' + \boldsymbol{\Psi}'_0)\cdot\boldsymbol{\Omega}_o] = 0 \tag{4}$$

It follows that necessary conditions for a steady rotation while the body is vibrating are: 1) mean and principal (deformation) axes must coincide; and 2) the diagonal element of $\boldsymbol{\Psi}'_q$ which corresponds to the axis of rotation must be zero. Rotation is then about the appropriate principal axis of the reference configuration.

The simple example used previously will illustrate these conditions. For the model of Fig. 1a, a steady rotation about the (fixed) $\bar{x}$-axis is possible for the vibrating beam. For the model of Fig. 2a, however, only symmetric modes are possible since then mean and principal axes coincide (Fig. 3a). For antisymmetric modes no such solution is possible and the beam must have an over-all small "wobble" superimposed on the steady rotation (Fig. 3b). The simple physical explanation is that, whereas the gyroscopic couples of the end disks cancel for the symmetric mode, they add for the antisymmetric mode.

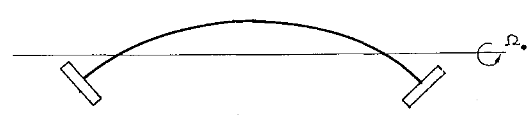

Fig. 3a Possible steady rotation.

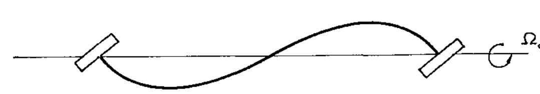

Fig. 3b Modulated steady rotation.

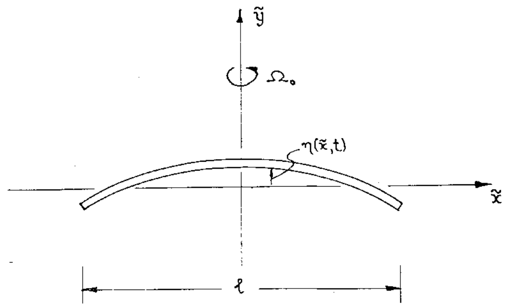

Fig. 4 Beam bending in plane of rotation vector.

3. The Elemental Equation of Equilibrium

If equations of motion are to be constructed for bodies having rapid rates of spin, it becomes essential to include the effect of relative rotation of the elements of the body in modifying the internal stress field. The classical equations of equilibrium, since they neglect rotation, are not sufficient for this purpose. Retaining the assumption of small strain but allowing for small rotation of body elements, the appropriate (nonlinear) equation of equilibrium is,[6]

$$\Delta m \times \{\text{acceleration of element}\} = \boldsymbol{\nabla}\cdot[\boldsymbol{\Sigma}\cdot(\mathcal{J} + \boldsymbol{\nabla}\mathbf{q})] \quad (5)$$

($\mathcal{J}$ = unit tensor) where $\boldsymbol{\Sigma}$ is the stress tensor. In applying this equation to bodies with equilibrium stress fields (as, for example, due to steady spin) the linearized form becomes

$$\Delta m \times \{\text{perturbation acceleration of element}\} = \boldsymbol{\nabla}\cdot[\bar{\boldsymbol{\Sigma}}\cdot\boldsymbol{\nabla}\mathbf{q}] + \boldsymbol{\nabla}\cdot\boldsymbol{\Sigma} \quad (6)$$

where $\bar{\boldsymbol{\Sigma}}$ is the equilibrium stress tensor satisfying

$$\Delta m \times \{\text{steady acceleration of element}\} = \boldsymbol{\nabla}\cdot\bar{\boldsymbol{\Sigma}} \quad (7)$$

and $\boldsymbol{\Sigma}$, $\mathbf{q}$ are perturbation stress and displacement fields, respectively. The equilibrium stress field is computed in the classical manner as for a massive body.[6] In applications wherein the effects of element rotation are significant, the rotations are larger than the strains and the gradient of $\mathbf{q}$, $\boldsymbol{\nabla}\mathbf{q}$, may usually be replaced in Eq. (6) by its antisymmetric part $\boldsymbol{\nabla}_a\mathbf{q}$.

Let $\mathbf{q}$ be the displacement field for free motion of the body referred to mean axes and let Q be a function space whose elements, defined in the domain of the body, have suitable continuity properties such that $\mathbf{q}\varepsilon Q$. Let $\mathbf{p}$ be any arbitrary element of Q. Then, if a steady rotational motion is possible in the sense of the previous section, the equation of motion of the body elements may be expressed in the integral form,

$$\int_V \frac{\delta^2\mathbf{q}}{\delta t^2}\cdot\mathbf{p}\,\Delta m + 2\boldsymbol{\Omega}_o\cdot\int_V \frac{\delta\mathbf{q}}{\delta t}\cdot\mathbf{p}\,\Delta m - \boldsymbol{\Omega}_o\cdot\int_V[\mathbf{q}\cdot\mathbf{p}\mathcal{J} - \mathbf{q}\mathbf{p}]\,\Delta m\cdot\boldsymbol{\Omega}_o - \int_V \bar{\boldsymbol{\Sigma}}:\ (\boldsymbol{\nabla}_a\mathbf{q}\cdot\boldsymbol{\nabla}_a\mathbf{p})\,\Delta V + \int_V \boldsymbol{\Sigma}(\mathbf{q})\cdot\boldsymbol{\nabla}_s\mathbf{p}\,\Delta V = 0 \quad (8)$$

where $\nabla_s\mathbf{p}$ is the symmetric part of $\nabla\mathbf{p}$ (the strain tensor) and where $\bar{\boldsymbol{\Sigma}}$ is the solution of

$$\boldsymbol{\nabla}\cdot\bar{\boldsymbol{\Sigma}} = \Delta m(\boldsymbol{\Omega}_0\boldsymbol{\Omega}_0 - \boldsymbol{\Omega}_0{}^2\mathcal{J})\cdot(\bar{\mathbf{r}} + \bar{\mathbf{q}}) \quad (9)$$

[here $\bar{\mathbf{q}}$ corresponds to Professor Ashley's "steady dilatation" solution, Eq. (57) of Ref. 1,

$$\sum_j \xi_j{}^d\boldsymbol{\Phi}_j$$

Taking

$$\mathbf{q} = \sum_i \xi_i{}^h\boldsymbol{\Phi}_i$$

and

$$\mathbf{p} = \boldsymbol{\Phi}_j$$

gives Eq. (59) of Ref. 1 with the addition of the term

$$\sum_i \xi_i{}^h \int_V \bar{\boldsymbol{\Sigma}}:(\boldsymbol{\nabla}_a\boldsymbol{\Phi}_i\cdot\boldsymbol{\nabla}_a\boldsymbol{\Phi}_j)\Delta V$$

(the sign of the third term of Eq. (56) appears to be incorrect—the sign of $\Omega_0{}^2$ in Eq. (59) is certainly at variance with that obtained by simple physical reasoning for a whirling shaft).

As a simple example of the importance of this term, consider a beam bending in the plane of rotation as shown in Fig. 4. The equilibrium stress tensor is simply

$$\bar{\boldsymbol{\Sigma}} = \int_{\tilde{x}}^{l/2} m(\tilde{x}')\,\Omega_0{}^2\tilde{x}'d\tilde{x}'\,\mathbf{ii}$$

where $m(\tilde{x})$ is the beam mass per unit length. If $\mathbf{q} = \eta(\tilde{x},t)\mathbf{j}$ represents bending of the beam, Eq. (8) yields

$$\int_{-l/2}^{l/2} m\frac{\partial^2\eta}{\partial t^2}\cdot\delta\eta\,d\tilde{x} - \Omega_o{}^2\int_{-l/2}^{l/2}\frac{\partial\eta}{\partial\tilde{x}}\cdot\frac{\partial\delta\eta}{\partial\tilde{x}}\int_{\tilde{x}}^{l/2} m(\tilde{x}') \times \tilde{x}'d\tilde{x}'d\tilde{x} + \int_{-l/2}^{l/2} EI\,\frac{\partial^2\eta}{\partial\tilde{x}^2}\cdot\frac{\partial^2\delta\eta}{\partial\tilde{x}^2}\,d\tilde{x} = 0 \quad (10)$$

The second and third terms of Eq. (8) give no contribution here since bending is restricted to the plane of rotation. Equation (10) implies the differential equation

$$m\,\frac{\partial^2\eta}{\partial t^2} + \frac{\partial^2}{\partial\tilde{x}^2}\left(EI\,\frac{\partial^2\eta}{\partial\tilde{x}^2}\right) + \Omega_o{}^2\left\{m(\tilde{x})\,\tilde{x}\,\frac{\partial\eta}{\partial\tilde{x}} - \frac{\partial^2\eta}{\partial\tilde{x}^2}\int_{\tilde{x}}^{l/} m(\tilde{x}')\tilde{x}'d\tilde{x}'\right\} = 0$$

(cf. Ref. 4, p. 96) together with a set of boundary conditions.

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Thrust Efficiencies of Electromagnetic Engines

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Nomenclature

$|e|$ = electronic charge, amp·sec
g = acceleration of gravity, m/sec²

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