Int. J. Solids Structures, 1966, Vol. 2, pp. 665 to 684. Pergamon Press Ltd. Printed in Great Britain

ELASTIC WAVE PROPAGATION IN HETEROGENEOUS PLATES

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Abstract—A two-dimensional linear theory of motions of heterogeneous plates is deduced from the three-dimensional theory of elasticity. Transverse shear deformations and rotatory inertia are included in the present general theory. The heterogeneity of the material is considered to be only in the thickness direction of the plate. The general plate theory is specialized to cases of symmetrically laminated aeolotropic, orthotropic and isotropic plates. A plate theory of the Kirchhoff type is also deduced. Frequency equations for the propagation of harmonic waves in an infinite two-layer isotropic plate in plane strain are obtained by the elasticity and the present plate theories. Several numerical examples are solved and their results are presented in graphs.

NOTATION

A_{ij}, B_{ij}, D_{ij}	constants defined by equation (15) or (16)
$\mathscr{A}_{ij}, \mathscr{B}_{ij}, \mathscr{D}_{ij}$	operators defined by equations (18)–(35)
a, b	thickness of top and bottom layer of two-layer plate
c	phase velocity ($= p/f$)
E_{ij}	elastic coefficients ($i, j = 1, 2, 3, 4, 5, 6$)
E	Young modulus
F_x, F_y, F_z	body forces
f	wave number ($= 2\pi/\Lambda$)
h	thickness of plate
$\bar{K}$	coefficient of modified shear modulus
L, L^0, H	functional operators
N, M, Q	plate-stresses defined by equations (7)–(9)
$\bar{p}_z$	traverse normal stress at $z = h$
p_i, P_i	forces defined by equation (45), $i = x, y, z$
R_0, R_1, R_2	constants defined by equation (46)
t	time
u, v, w	displacement components in x, y, z directions
u^0, v^0, w^0	displacement components at $z = 0$
x, y, z	Cartesian coordinates
α, β	defined by equation (A14-15)
γ, δ, e, k	plate parameters defined by equation (A18-21)
κ	curvature component defined by equation (13)
ε	strain component defined by equation (10)
ε^0	strain component at $z = 0$ defined by equation (12)
τ	stress components
ψ_x, ψ_y	slope functions defined by equations (4), (5)
F	wave parameter
Λ	wave length
λ, μ	Lamé constants
ν	Poisson ratio
ρ	density
$(\ )_{,i}$	partial differentiation with respect to $i (i = x, y, t)$

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1. INTRODUCTION

IN RECENT YEARS some interest arose towards the development of elasticity theories for plates that are non-homogeneous in the thickness direction. The elastostatic bending and stretching theory for such plates, based on the Euler–Bernoulli hypothesis, was established in [1] and [2].

In the present study the equations of motion are established for heterogeneous anisotropic plates incorporating the effect of shear deformations in a way suggested by Hencky [3], Uflyand [4] and Mindlin [5] for homogeneous plates. Associated with the system of plate equations there are stated suitable boundary and initial conditions to ensure a unique solution.

It will be shown that the plate heterogeneity introduces a coupling phenomenon between bending and stretching of the type found by Reissner and Stavsky [1, 2] for the static case.

The general theory is applied to the propagation of plane strain waves of the Rayleigh–Lamb type [6, 7] in specific two-layer isotropic plates.

2. FORMULATION OF PLATE THEORY

Let us consider a thin elastic heterogeneous plate of thickness h, referred to an x, y, z system of Cartesian coordinates. The lower and upper surfaces of the plate are $z = 0, h$ and its cylindrical boundaries $f_b(x, y) = 0$ are defined by plane curves parallel to the x–y plane. The faces of the plate are assumed to be free of shear stresses but subjected to transverse normal stress, as follows

$$\tau_{xz}|_{z=0,h} = 0, \qquad \tau_{yz}|_{z=0,h} = 0 \tag{1}$$

$$\tau_z|_{z=0} = 0, \qquad \tau_z|_{z=h} = \bar{p}_z. \tag{2}$$

The non-homogeneity of the plate is only in the thickness direction z and it may be of two types: (i) the elastic moduli vary continuously in the z direction of the so called "heterogeneous plate", (ii) thin homogeneous layers of different elastic properties are composed to form a "laminated plate" in which the moduli are step functions of z.

Assume the following general Hooke's law for the stress–strain relations, wherein each stress component is a linear function of all six strain components

$$\begin{bmatrix} \tau_x \\ \tau_y \\ \tau_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} & E_{15} & E_{16} \\ & E_{22} & E_{23} & E_{24} & E_{25} & E_{26} \\ & & E_{33} & E_{34} & E_{35} & E_{36} \\ & & & E_{44} & E_{45} & E_{46} \\ \text{Symmetric} & & & & E_{55} & E_{56} \\ \text{Matrix } E & & & & & E_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix} \tag{3}$$

The twenty-one elastic coefficients E_{ij} are specified functions of z but do not vary in the x, y directions.

In order to account for transverse shear deformation and rotary inertia effects in the heterogeneous plate theory to be established, we follow Hencky [3] and assume the displacement components to be of the form

$$u(x, y, z, t) = u^0(x, y, t) + z\psi_x(x, y, t) \tag{4}$$

$$v(x, y, z, t) = v^0(x, y, t) + z\psi_y(x, y, t) \tag{5}$$

$$w(x, y, z, t) = w^0(x, y, t). \tag{6}$$

Note that these relations involve combined action of bending and extension which characterizes the behavior of heterogeneous plates as shown by Reissner and Stavsky [1, 2].

Defining stress resultants, stress couples, reference surface strains, bending curvatures and transverse shear strains the following relations are obtained in view of equations (4)–(6),

$$(N_x, N_y, N_{xy}) = \int_0^h (\tau_x, \tau_y, \tau_{xy})\, dz \tag{7}$$

$$(Q_x, Q_y) = \int_0^h (\tau_{xz}, \tau_{yz})\, dz \tag{8}$$

$$(M_x, M_y, M_{xy}) = \int_0^h (\tau_x, \tau_y, \tau_{xy}) z\, dz \tag{9}$$

$$(\varepsilon_x, \varepsilon_y, \varepsilon_{xy}) = (\varepsilon_x^0, \varepsilon_y^0, \varepsilon_{xy}^0) + z(\kappa_x, \kappa_y, \kappa_{xy}) \tag{10}$$

$$\varepsilon_z = 0; \qquad \varepsilon_{yz} = \varepsilon_{yz}^0 = w^0{}_{,y} + \psi_y; \qquad \varepsilon_{xz} = \varepsilon_{xz}^0 = w^0{}_{,x} + \psi_x \tag{11}$$

$$\varepsilon_x^0 = u^0{}_{,x}; \qquad \varepsilon_y^0 = v^0{}_{,y}; \qquad \varepsilon_{xy}^0 = u^0{}_{,y} + v^0{}_{,x} \tag{12}$$

$$\kappa_x = \psi_{x,x}; \qquad \kappa_y = \psi_{y,y}; \qquad \kappa_{xy} = \psi_{x,y} + \psi_{y,x}. \tag{13}$$

In order to obtain plate stress–strain relations expressions (10)–(13) are introduced into (3) and the results are integrated according to the definitions (7)–(9) to give

$$\begin{bmatrix} N_x \\ N_y \\ Q_y \\ Q_x \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} = \left[\begin{array}{ccccc|ccc} A_{11} & A_{12} & A_{14} & A_{15} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{24} & A_{25} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{14} & A_{24} & A_{44} & A_{45} & A_{46} & B_{14} & B_{24} & B_{46} \\ A_{15} & A_{25} & A_{45} & A_{55} & A_{56} & B_{15} & B_{25} & B_{56} \\ A_{16} & A_{26} & A_{46} & A_{56} & A_{66} & B_{16} & B_{26} & B_{66} \\ \hline B_{11} & B_{12} & B_{14} & B_{15} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{24} & B_{25} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{46} & B_{56} & B_{66} & D_{16} & D_{26} & D_{66} \end{array}\right] \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \varepsilon_{yz}^0 \\ \varepsilon_{xz}^0 \\ \varepsilon_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} \tag{14}$$

where the constants A, B and D are defined by the following integrals for a continuously heterogeneous plate

$$(A_{ij}, B_{ij}, D_{ij}) = \int_0^h E_{ij}(1, z, z^2)\,\mathrm{d}z \qquad (i,j = 1,2,4,5,6). \tag{15}$$

For a n-layer laminated plate these relations become

$$(A_{ij}, B_{ij}, D_{ij}) = \sum_{m=1}^{n} \int_{h_{m-1}}^{h_m} E_{ij}^m(1, z, z^2)\,\mathrm{d}z \qquad (i,j = 1,2,4,5,6) \tag{16}$$

where E_{ij}^m are the moduli of the homogeneous layer m of thickness $h_m - h_{m-1}$.

Substituting equations (11)–(13) into equation (14) the stress resultants and couples are then expressed in terms of the displacement components u^0, v^0, w^0 of the reference plane, and the components ψ_x, ψ_y of change of slope of the normal to the undeformed reference plane as follows

$$\begin{bmatrix} N_x \\ N_y \\ Q_y \\ Q_x \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} \mathscr{A}_{11} & \mathscr{A}_{12} & \mathscr{A}_{13} & A_{15}+\mathscr{B}_{11} & A_{14}+\mathscr{B}_{12} \\ \mathscr{A}_{21} & \mathscr{A}_{22} & \mathscr{A}_{23} & A_{25}+\mathscr{B}_{21} & A_{24}+\mathscr{B}_{22} \\ \mathscr{A}_{41} & \mathscr{A}_{42} & \mathscr{A}_{43} & A_{45}+\mathscr{B}_{41} & A_{44}+\mathscr{B}_{42} \\ \mathscr{A}_{51} & \mathscr{A}_{52} & \mathscr{A}_{53} & A_{55}+\mathscr{B}_{51} & A_{45}+\mathscr{B}_{52} \\ \mathscr{A}_{61} & \mathscr{A}_{62} & \mathscr{A}_{63} & A_{65}+\mathscr{B}_{61} & A_{46}+\mathscr{B}_{62} \\ \mathscr{B}_{11} & \mathscr{B}_{12} & \mathscr{B}_{13} & B_{15}+\mathscr{D}_{11} & B_{14}+\mathscr{D}_{12} \\ \mathscr{B}_{21} & \mathscr{B}_{22} & \mathscr{B}_{23} & B_{25}+\mathscr{D}_{21} & B_{24}+\mathscr{D}_{22} \\ \mathscr{B}_{61} & \mathscr{B}_{62} & \mathscr{B}_{63} & B_{56}+\mathscr{D}_{61} & B_{46}+\mathscr{D}_{62} \end{bmatrix} \begin{bmatrix} u^0 \\ v^0 \\ w^0 \\ \psi_x \\ \psi_y \end{bmatrix} \tag{17}$$

where

$$(\mathscr{A}_{11}, \mathscr{B}_{11}) = (A_{11}, B_{11})(\ )_{,x} + (A_{16}, B_{16})(\ )_{,y} \tag{18}$$

$$(\mathscr{A}_{12}, \mathscr{B}_{12}) = (A_{16}, B_{16})(\ )_{,x} + (A_{12}, B_{12})(\ )_{,y} \tag{19}$$

$$(\mathscr{A}_{13}, \mathscr{B}_{13}) = (A_{15}, B_{15})(\ )_{,x} + (A_{14}, B_{14})(\ )_{,y} \tag{20}$$

$$(\mathscr{A}_{21}, \mathscr{B}_{21}) = (A_{12}, B_{12})(\ )_{,x} + (A_{26}, B_{26})(\ )_{,y} \tag{21}$$

$$(\mathscr{A}_{22}, \mathscr{B}_{22}) = (A_{26}, B_{26})(\ )_{,x} + (A_{22}, B_{22})(\ )_{,y} \tag{22}$$

$$(\mathscr{A}_{23}, \mathscr{B}_{23}) = (A_{25}, B_{25})(\ )_{,x} + (A_{24}, B_{24})(\ )_{,y} \tag{23}$$

$$(\mathscr{A}_{41}, \mathscr{B}_{41}) = (A_{14}, B_{14})(\ )_{,x} + (A_{46}, B_{46})(\ )_{,y} \tag{24}$$

$$(\mathscr{A}_{42}, \mathscr{B}_{42}) = (A_{46}, B_{46})(\ )_{,x} + (A_{24}, B_{24})(\ )_{,y} \tag{25}$$

$$\mathscr{A}_{43} = A_{45}(\ )_{,x} + A_{44}(\ )_{,y} \tag{26}$$

$$(\mathscr{A}_{51}, \mathscr{B}_{51}) = (A_{15}, B_{15})(\ )_{,x} + (A_{56}, B_{56})(\ )_{,y} \tag{27}$$

$$(\mathscr{A}_{52}, \mathscr{B}_{52}) = (A_{56}, B_{56})(\ )_{,x} + (A_{25}, B_{25})(\ )_{,y} \tag{28}$$

$$\mathscr{A}_{53} = A_{55}(\ )_{,x} + A_{45}(\ )_{,y} \tag{29}$$

$$(\mathscr{A}_{61}, \mathscr{B}_{61}) = (A_{16}, B_{16})(\ )_{,x} + (A_{66}, B_{66})(\ )_{,y} \tag{30}$$

$$(\mathscr{A}_{62}, \mathscr{B}_{62}) = (A_{66}, B_{66})(\ )_{,x} + (A_{26}, B_{26})(\ )_{,y} \tag{31}$$

$$(\mathscr{A}_{63}, \mathscr{B}_{63}) = (A_{56}, B_{56})(\ )_{,x} + (A_{46}, B_{46})(\ )_{,y} \tag{32}$$

$$(\mathscr{D}_{11}, \mathscr{D}_{12}) = (D_{11}, D_{16})(\ )_{,x} + (D_{16}, D_{12})(\ )_{,y} \tag{33}$$

$$(\mathscr{D}_{21}, \mathscr{D}_{22}) = (D_{12}, D_{26})(\ )_{,x} + (D_{26}, D_{22})(\ )_{,y} \tag{34}$$

$$(\mathscr{D}_{61}, \mathscr{D}_{62}) = (D_{16}, D_{66})(\ )_{,x} + (D_{66}, D_{26})(\ )_{,y} \tag{35}$$

The two-dimensional theory of extensional and flexural motions of heterogeneous anisotropic plates is deduced from the dynamical equations of three-dimensional elasticity

$$\tau_{x,x} + \tau_{xy,y} + \tau_{xz,z} + F_x = \rho u_{,tt} \tag{36}$$

$$\tau_{xy,x} + \tau_{y,y} + \tau_{yz,z} + F_y = \rho v_{,tt} \tag{37}$$

$$\tau_{xz,x} + \tau_{yz,y} + \tau_{z,z} + F_z = \rho w_{,tt} \tag{38}$$

where $\rho = \rho(x, y, z)$ is the material density and F_i is the body force i-axis component. These equations are converted to plate-stress equations of motion by the method of Boussinesq [8], first they are integrated over the plate thickness and then equations (36) and (37) are multiplied by z and integrated over the thickness. Making use of equations (1) through (9) the following equations of motion are obtained

$$N_{x,x} + N_{xy,y} + p_x = R_0 u^0_{,tt} + R_1 \psi_{x,tt} \tag{39}$$

$$N_{xy,x} + N_{y,y} + p_y = R_0 v^0_{,tt} + R_1 \psi_{y,tt} \tag{40}$$

$$Q_{x,x} + Q_{y,y} + p_z = R_0 w^0_{,tt} \tag{41}$$

$$M_{x,x} + M_{xy,y} - Q_x = R_1 u^0_{,tt} + R_2 \psi_{x,tt} - P_x \tag{42}$$

$$M_{xy,x} + M_{y,y} - Q_y = R_1 v^0_{,tt} + R_2 \psi_{y,tt} - P_y \tag{43}$$

where the constants R_i are of the form

$$(R_0, R_1, R_2) = \int_0^h \rho(1, z, z^2)\,dz \tag{44}$$

for continuously heterogeneous plates with $\rho = \rho(z)$, and

$$(p_x, p_y, p_z, -\bar{p}_z) = \int_0^h (F_x, F_y, F_z)\,dz; \qquad (P_x, P_y) = \int_0^h (F_x, F_y) z\,dz. \tag{45}$$

In case of layered plates expressions (44) read

$$(R_0, R_1, R_2) = \sum_{m=1}^{n} \int_{h_{m-1}}^{h_m} \rho^m (1, z, z^2)\,dz \tag{46}$$

where ρ^m is the density of layer m.

The five equations of motion (39) to (43) are further expressed in terms of u^0, v^0, w^0, ψ_x and ψ_y using equations (17) with the result

$$L_{11}u^0 + L_{12}v^0 + L_{13}w^0 + L_{14}\psi_x + L_{15}\psi_y + p_x = 0 \tag{47}$$

$$L_{21}u^0 + L_{22}v^0 + L_{23}w^0 + L_{24}\psi_x + L_{25}\psi_y + p_y = 0 \tag{48}$$

$$L_{31}u^0 + L_{32}v^0 + L_{33}w^0 + L_{34}\psi_x + L_{35}\psi_y + p_z = 0 \tag{49}$$

$$L_{41}u^0 + L_{42}v^0 + L_{43}w^0 + L_{44}\psi_x + L_{45}\psi_y + P_x = 0 \tag{50}$$

$$L_{51}u^0 + L_{52}v^0 + L_{53}w^0 + L_{54}\psi_x + L_{55}\psi_y + P_y = 0. \tag{51}$$

The functional operators L_{ij} are of the form

$$L_{11} = A_{11}(\ )_{,xx}+6A_{16}(\ )_{,xy}+A_{66}(\ )_{,yy}-R_0(\ )_{,tt} \tag{52}$$

$$L_{12} = L_{21} = A_{16}(\ )_{,xx}+(A_{12}+A_{66})(\ )_{,xy}+A_{26}(\ )_{,yy} \tag{53}$$

$$L_{13} = L_{31} = A_{15}(\ )_{,xx}+(A_{14}+A_{56})(\ )_{,xy}+A_{46}(\ )_{,yy} \tag{54}$$

$$L_{14}, L_{41} = \pm A_{15}(\ )_{,x}\pm A_{56}(\ )_{,y}+B_{11}(\ )_{,xx}+2B_{16}(\ )_{,xy}+B_{66}(\ )_{,yy}-R_1(\ )_{,tt} \tag{55}$$

$$L_{15}, L_{51} = \pm A_{14}(\ )_{,x}\pm A_{46}(\ )_{,y}+B_{16}(\ )_{,xx}+(B_{12}+B_{66})(\ )_{,xy}+B_{26}(\ )_{,yy} \tag{56}$$

$$L_{22} = A_{66}(\ )_{,xx}+2A_{26}(\ )_{,xy}+A_{22}(\ )_{,yy}-R_0(\ )_{,tt} \tag{57}$$

$$L_{23} = L_{32} = A_{56}(\ )_{,xx}+(A_{25}+A_{46})(\ )_{,xy}+A_{24}(\ )_{,yy} \tag{58}$$

$$L_{24}, L_{42} = \pm A_{56}(\ )_{,x}\pm A_{25}(\ )_{,y}+B_{16}(\ )_{,xx}+(B_{12}+B_{66})(\ )_{,xy}+B_{26}(\ )_{,yy} \tag{59}$$

$$L_{25}, L_{52} = \pm A_{46}(\ )_{,x}\pm A_{24}(\ )_{,y}+B_{66}(\ )_{,xx}+2B_{26}(\ )_{,xy}+B_{22}(\ )_{,yy}-R_1(\ )_{,tt} \tag{60}$$

$$L_{33} = A_{55}(\ )_{,xx}+2A_{45}(\ )_{,xy}+A_{44}(\ )_{,yy}-R_0(\ )_{,tt} \tag{61}$$

$$L_{34}, L_{43} = \pm A_{55}(\ )_{,x}\pm A_{45}(\ )_{,y}+B_{51}(\ )_{,xx}+(B_{14}+B_{56})(\ )_{,xy}+B_{46}(\ )_{,yy} \tag{62}$$

$$L_{35}, L_{53} = \pm A_{45}(\ )_{,x}\pm A_{44}(\ )_{,y}+B_{56}(\ )_{,xx}+(B_{25}+B_{46})(\ )_{,xy}+B_{24}(\ )_{,yy} \tag{63}$$

$$L_{44} = -A_{55}+D_{11}(\ )_{,xx}+2D_{16}(\ )_{,xy}-D_{66}(\ )_{,yy}-R_2(\ )_{,tt} \tag{64}$$

$$L_{45}, L_{54} = -A_{45}\pm(B_{14}-B_{56})(\ )_{,x}\pm(B_{46}-B_{25})(\ )_{,y}+D_{16}(\ )_{,xx} +(D_{12}+D_{66})(\ )_{,xy}+D_{26}(\ )_{,yy} \tag{65}$$

$$L_{55} = -A_{44}+D_{66}(\ )_{,xx}+2D_{26}(\ )_{,xy}+D_{22}(\ )_{,yy}-R_2(\ )_{,tt}. \tag{66}$$

The tenth-order system (47) through (51) is the main contribution of the present theory. It shows that extensional and flexural motions are generally coupled in a heterogeneous plate and u^0, v^0, w^0 and ψ_x, ψ_y are to be determined simultaneously.

Appropriate initial and boundary conditions which are sufficient to assure a unique solution of the plate equations (47)–(51), are as follows

(i) Initial values of $u_n^0, u_s^0, w^0, \psi_n, \psi_s$ and their time derivatives, throughout the plate.

(ii) Any combination of the following boundary conditions along an edge $f_b(x, y) = 0$

$$u_n = \bar{u}_n \quad \text{or} \quad N_n = \bar{N}_n \tag{67}$$

$$u_s^0 = \bar{u}_s^0 \quad \text{or} \quad N_{ns} = \bar{N}_{ns} \tag{68}$$

$$w^0 = \bar{w}^0 \quad \text{or} \quad Q_n = \bar{Q}_n \tag{69}$$

$$\psi_n = \bar{\psi}_n \quad \text{or} \quad M_n = \bar{M}_n \tag{70}$$

$$\psi_s = \bar{\psi}_s \quad \text{or} \quad M_{ns} = \bar{M}_{ns} \tag{71}$$

where the barred quantities are prescribed.

(iii) On the plate boundaries $z = 0, h$ there are specified the transverse deflection w^0 or the transverse load $\bar{p}_z$.

3. KIRCHHOFF'S TYPE PLATE THEORY

It is of interest, in many instances, to resort to Kirchhoff's hypothesis when analyzing heterogeneous plates. The components ψ_x, ψ_y of change of slope of the normal to the undeformed reference plane are now related to the transverse deflection w^0 in the following manner

$$\psi_x = -w^0{}_{,x}; \qquad \psi_y = -w^0{}_{,y} \tag{72}$$

Consequently, the five equilibrium equations (39) to (43), in view of equations (10)–(14), can be reduced to three simultaneous equations for u^0, v^0, w^0 in the form

$$L^0_{11}u^0 + L^0_{12}v^0 + L^0_{13}w^0 + p_x = 0 \tag{73}$$

$$L^0_{21}w^0 + L^0_{22}v^0 + L^0_{23}w^0 + p_y = 0 \tag{74}$$

$$L^0_{31}u^0 + L^0_{32}v^0 + L^0_{33}w^0 + p_z = 0. \tag{75}$$

The functional operators L^0_{ij} are given by the following expressions

$$L^0_{12} = L^0_{21} = L_{12}, \qquad L^0_{22} = L_{22} \tag{76}$$

$$L^0_{13} = -L^0_{31} = -B_{11}(\;)_{,xxx} - 3B_{16}(\;)_{,xxy} - (B_{12}+2B_{66})(\;)_{,xyy} - B_{26}(\;)_{,yyy} + R_1(\;)_{,xtt} \tag{77}$$

$$L^0_{23} = -L^0_{23} = -B_{16}(\;)_{,xxx} - (B_{12}+2B_{16})(\;)_{,xxy} - 3B_{26}(\;)_{,xyy} - B_{22}(\;)_{,yyy} + R_1(\;)_{,ytt} \tag{78}$$

$$L^0_{33} = -D_{11}(\;)_{,xxxx} - 4D_{16}(\;)_{,xxxy} - (2D_{12}+4D_{66})(\;)_{,xxyy} - 4D_{26}(\;)_{,xyyy} - D_{22}(\;)_{,yyyy}$$
$$- R_0(\;)_{,tt} + R_2(\;)_{,xxtt} + R_2(\;)_{,yytt}. \tag{79}$$

The eighth-order system of equations (73)–(75) is qualitatively different from the tenth-order system (47)–(51) formulating a heterogeneous plate theory which abandons Kirchhoff's hypothesis. The reduction in the order of the differential equations is also reflected in the form and number of initial and boundary conditions which are now as follows:

(i) Throughout the plate: initial values of u^0_n, u^0_s, w^0 and their time derivatives.

(ii) Along an edge of the plate: any combination of the following boundary conditions

$$u^0 = \bar{u}^0 \quad \text{or} \quad N_n = \bar{N}_n \tag{80}$$

$$u^0_s = \bar{u}^0_s \quad \text{or} \quad N_{ns} = \bar{N}_{ns} \tag{81}$$

$$w^0 = \bar{w}^0 \quad \text{or} \quad Q_n + M_{ns,s} = R_n = \bar{R}_n \tag{82}$$

$$\psi_n = \bar{\psi}_n \quad \text{or} \quad M_n = \bar{M}_n. \tag{83}$$

(iii) On the plate faces $z = 0, h$: either $\bar{p}_z$ or w^0 are prescribed.

Note that the present eighth-order theory neglects the effect of transverse shear deformation but includes the effects of "coupled" and rotary inertia. If the plate behavior is independent of time the static equations pertain, they are of the general form (73)–(75) where the L^0's do not contain any R terms and u^0, v^0, w^0 are functions of x and y only. These three simultaneous elastostatic equations for the displacement components may be considered as an alternate formulation of the plate theory given by Reissner and Stavsky [1, 2], in terms of w^0 and an Airy stress function F.

4. SYMMETRICALLY HETEROGENEOUS ANISOTROPIC PLATES

An interesting special class of laminated plates derived from the general theory, formulated in equations (47)–(51), is the following:

Let the plate described in Section 2 be the upper half of a heterogeneous plate with total thickness h. For such a plate it is natural to locate the reference plane $z = 0$ at its midsurface, then the elastic moduli follow the symmetry law

$$E_{ij}(x, y, -z) = E_{ij}(x, y, +z). \tag{84}$$

Consequently

$$B_{ij} = \int_{-h/2}^{+h/2} E_{ij} z \, dz = 0, \qquad R_1 = \int_{-h/2}^{+h/2} \rho z \, dz = 0 \tag{85}$$

which symmetrize and simplify the functional operators L_{ij} but does not alter the general form of the system (47)–(51). This means that the coupling between extensional and flexural vibrations still exists but it stems now from the *anisotropy* of the plate material and not from its heterogeneity.

In case the symmetrical heterogeneous plate is monoclinic, some of the elastic moduli in (3) vanish

$$E_{14} = E_{15} = E_{24} = E_{25} = E_{34} = E_{35} = E_{46} = E_{56} = 0 \tag{86}$$

and consequently certain functional operators in (47)–(51) disappear and others are simplified as follows

$$L_{13} = L_{14} = L_{15} = L_{23} = L_{24} = L_{25} = 0, \qquad L_{ji} = L_{ij}. \tag{87}$$

The general plate equations are reduced to an extensional system

$$L_{11}u^0 + L_{12}v^0 + p_x = 0 \tag{88}$$

$$L_{12}u^0 + L_{22}v^0 + p_y = 0 \tag{89}$$

and a flexural system that are uncoupled,

$$L_{33}w^0 + L_{34}\psi_x + L_{35}\psi_y + p_z = 0 \tag{90}$$

$$L_{34}w^0 + L_{44}\psi_x + L_{45}\psi_y = 0 \tag{91}$$

$$L_{35}w^0 + L_{45}\psi_x + L_{55}\psi_y = 0 \tag{92}$$

where the L's are given by equations (52)–(66) after taking notice of equations (86) and (87). The elastodynamics equations for symmetrically orthotropic or isotropic plates, including shear and inertia terms, will be of the same form as for the monoclinic plate, equations (88)–(92), hence extensional and flexural vibrations will be uncoupled. Some simplifications will arise in the expressions for the L operators when appropriate stress–strain relations for orthotropic or isotropic plates are introduced in equations (52)–(66).

It is interesting to note that the equations of motion (88)–(89) and (90)–(92) for heterogeneous orthotropic and isotropic plates have the same form of Mindlin's [9, 5] equations for the corresponding homogeneous plates, the difference is only in the constants of the functional operators L.

The initial and boundary conditions associated with the extensional system (88), (89) are:

(i) Initial values of u_n^0, u_s^0 and their time derivative throughout the plate.

(ii) Any combination of the following boundary conditions on an edge $f_b(x, y) = 0$

$$u_n^0 = \bar{u}_n^0 \quad \text{or} \quad N_n = \bar{N}_n \tag{93}$$

$$u_s^0 = \bar{u}_s^0 \quad \text{or} \quad N_{ns} = \bar{N}_{ns}. \tag{94}$$

For the conditions adjoined to the flexural system (90)–(92) one writes

(i) Initial values of w^0, ψ_n, ψ_s and their time derivatives throughout the plate.

(ii) On an edge $f_b(x, y) = 0$

$$w^0 = \bar{w}^0 \quad \text{or} \quad Q_n = \bar{Q}_n \tag{95}$$

$$\psi_n = \bar{\psi}_n \quad \text{or} \quad M_n = \bar{M}_n \tag{96}$$

$$\psi_s = \bar{\psi}_s \quad \text{or} \quad M_{ns} = \bar{M}_{ns} \tag{97}$$

5. x–t STRAIN DEPENDANCE IN HETEROGENEOUS ANISOTROPIC PLATES

Let ε_y vanish and let both the shear strain components ε_{xy}, ε_{zy}, and the body force p_y be independent of the y-coordinate. The displacement components (4)–(6) take now the form

$$u(x, z, t) = u^0(x, t) + z\psi_x(x, t) \tag{98}$$

$$v(x, z, t) = v^0(x, t) + z\psi_y(x, t) \tag{99}$$

$$w(x, z, t) = w^0(x, t) \tag{100}$$

which upon introduction into the equilibrium equations (47)–(51) gives the same *tenth-order* system that contains now x and t as the independent variables.

The functional operators L_{ij} are therefore modified as follows

$$L_{11} = A_{11}(\ )_{,xx} - R_0(\ )_{,tt}; \qquad L_{12} = L_{21} = A_{16}(\ )_{,xx} \qquad (101), (102)$$

$$L_{13} = L_{31} = A_{15}(\ )_{,xx} \tag{103}$$

$$L_{14}, L_{41} = \pm A_{15}(\ )_{,x} + B_{11}(\ )_{,xx} - R_1(\ )_{,tt} \tag{104}$$

$$L_{15}, L_{51} = \pm A_{14}(\ )_{,x} + B_{16}(\ )_{,xx} \tag{105}$$

$$L_{22} = A_{66}(\ )_{,xx} - R_0(\ )_{,tt}; \qquad L_{23} = L_{32} = A_{56}(\ )_{,xx} \qquad (106), (107)$$

$$L_{24}, L_{42} = \pm A_{56}(\ )_{,x} + B_{16}(\ )_{,xx}; \qquad L_{25}, L_{52} = \pm A_{46}(\ )_{,x} + B_{66}(\ )_{,xx} - R_1(\ )_{,tt} \qquad (108), (109)$$

$$L_{33} = A_{55}(\ )_{,xx} - R_0(\ )_{,tt}; \qquad L_{34}, L_{43} = \pm A_{55}(\ )_{,x} + B_{51}(\ )_{,xx} \qquad (110), (111)$$

$$L_{35}, L_{53} = \pm A_{45}(\ )_{,x} + B_{56}(\ )_{,xx}; \qquad L_{44} = -A_{55} + D_{11}(\ )_{,xx} - R_2(\ )_{,tt} \qquad (112), (113)$$

$$L_{45}, L_{54} = -A_{45} \pm (B_{14} - B_{56})(\ )_{,x} + D_{16}(\ )_{,xx} \tag{114}$$

$$L_{55} = -A_{44} + D_{66}(\ )_{,xx} - R_2(\ )_{,tt}. \tag{115}$$

The tenth-order x–t dependent system (47)–(51) shows that even for the one-dimensional deformation case extensional and flexural motions are coupled for non-homogeneous plates composed of the most general anisotropic material. The appropriate initial and boundary conditions are

(i) Initial values of $u^0, v^0, w^0, \psi_x, \psi_y$ and their time derivative throughout the plate.
(ii) Boundary conditions on the edges $x = 0, 1$

$$u^0 = \bar{u}^0 \quad \text{or} \quad N_x = \bar{N}_x \tag{116}$$

$$v^0 = \bar{v}^0 \quad \text{or} \quad N_{xy} = \bar{N}_{xy} \tag{117}$$

$$w^0 = \bar{w}^0 \quad \text{or} \quad Q_x = \bar{Q}_x \tag{118}$$

$$\psi_x = \bar{\psi}_x \quad \text{or} \quad M_x = \bar{M}_x \tag{119}$$

$$\psi_y = \bar{\psi}_y \quad \text{or} \quad M_{xy} = \bar{M}_{xy}. \tag{120}$$

(iii) On the plate boundaries $z = 0, h: w^0$ or $\bar{p}_z$ are specified.

Similarly, a Kirchhoff type theory can be formulated when equations (98)–(110) are modified in view of equation (72) to read

$$u(x, z, t) = u^0(x, t) - zw^0(x, t)_{,x} \tag{121}$$

$$v(x, z, t) = v^0(x, t) \tag{122}$$

$$w(x, z, t) = w^0(x, t) \tag{123}$$

which upon introduction into equations (73)–(75) results with the same *eighth*-order system with x and t as independent variables. The L^0's are consequently reduced to the following form

$$L_{11}^0 = A_{11}(\ )_{,xx} - R_0(\ )_{,tt}; \qquad L_{12}^0 = L_{21}^0 = A_{16}(\ )_{,xx} \tag{124), (125}$$

$$L_{13}^0 = -L_{31}^0 = -B_{11}(\ )_{,xxx} + R_1(\ )_{,xtt} \tag{126}$$

$$L_{22}^0 = A_{66}(\ )_{,xx} - R_0(\ )_{,tt}; \qquad L_{23}^0 = -L_{32}^0 = -B_{16}(\ )_{,xxx} \tag{127), (128}$$

$$L_{33}^0 = -D_{11}(\ )_{,xxxx} - R_0(\ )_{,tt} + R_2(\ )_{,xxtt}. \tag{129}$$

It is noted that the displacement components remain coupled even for the one-dimensional case due to the anisotropy of the considered plate. The form of the boundary conditions along $x = 0,1$ will be of the form given in equations (80)–(83) and the remaining conditions will be of the type explained for the Kirchhoff's type two-dimensional theory.

6. x–t STRAIN DEPENDENCE IN HETEROGENEOUS ORTHOTROPIC PLATES

An interesting reduction occurs in the one-dimensional equations, given in Section 5, when the plate material is orthotropic. Equations (47)–(51) are separated to a sixth-order system for u^0, w^0 and ψ_x, and a fourth-order system for v^0 and ψ_y with the result

$$L_{11}u^0 + L_{14}\psi_x + p_x = 0 \tag{130}$$

$$L_{33}w^0 + L_{34}\psi_x + p_z = 0 \tag{131}$$

$$L_{41}u^0 + L_{43}w^0 + L_{44}\psi_x + P_x = 0 \tag{132}$$

and

$$L_{22}v^0 + L_{25}\psi_y + p_y = 0 \tag{133}$$

$$L_{52}v^0+L_{55}\psi_y+P_y = 0 \quad (134)$$

where

$$L_{11} = A_{11}(\ )_{,xx}-R_0(\ )_{,tt}; \qquad L_{14} = L_{41} = B_{11}(\ )_{,xx}-R_1(\ )_{,tt} \quad (136)$$

$$L_{33} = A_{55}(\ )_{,xx}-R_0(\ )_{,tt}; \qquad L_{34}, L_{43} = \pm A_{55}(\ )_{,x} \quad (137), (138)$$

$$L_{44} = -A_{55}+D_{11}(\ )_{,xx}-R_2(\ )_{,tt} \quad (139)$$

and

$$L_{22} = A_{66}(\ )_{,xx}; \qquad L_{25}, L_{52} = B_{66}(\ )_{,xx}-R_1(\ )_{,tt} \quad (140), (141)$$

$$L_{55} = -A_{44}+D_{66}(\ )_{,xx}-R_2(\ )_{,tt}. \quad (142)$$

The boundary conditions associated with the system (130)–(132) are given by equations (116), (118), (119) while the conditions (117) and (120) must be satisfied by the simultaneous equations (133), (134).

A one-dimensional Kirchhoff-type theory for orthotropic heterogeneous plates is obtained from the eighth-order system (73)–(75) when appropriate simplifications are introduced into the L^0's. There results a sixth-order system for u^0 and w^0 and a separate second-order equation for v^0,

$$L_{11}^0 u^0+L_{13}^0 w^0+p_x = 0 \quad (143)$$

$$L_{31}^0 u^0+L_{33}^0 w^0+p_z = 0 \quad (144)$$

and

$$L_{22}^0 v^0+p_y = 0 \quad (145)$$

where

$$L_{13}^0 = -L_{31}^0 = -B_{11}(\ )_{,xxx}+R_1(\ )_{,xtt} \quad (146)$$

$$L_{33}^0 = -D_{11}(\ )_{,xxxx}+R_0(\ )_{,tt}+R_2(\ )_{,xxtt} \quad (147)$$

L_{11}^0 is given by equation (102) and L_{22}^0 by (106).

Three boundary conditions, along $x = 0,1$, are associated with the sixth-order system (143), (144), one of which is the same as (116), and

$$w^0 = \bar{w}^0 \quad \text{or} \quad R_x = \bar{R}_x$$
$$w^0{}_{,x} = \bar{w}^0{}_{,x} \quad \text{or} \quad M_x = \bar{M}_x. \quad (148)$$

Correspondingly, the simple boundary condition to be satisfied by v^0 on $x = 0,1$ is

$$v^0 = \bar{v}^0 \quad \text{or} \quad N_{xy} = \bar{N}_{xy} \quad (149)$$

Equations (143), (144) are remarkable for the coupling of u^0 and w^0 which disappears only when B_{11} and R_1 vanish. Then equations (143), (144) reduce to the classical equations of longitudinal and flexural vibrations, respectively, of homogeneous rods (see, e.g., Kolsky's [10] monograph).

7. x–t STRAIN DEPENDENCE IN ISOTROPIC TWO-LAYER PLATES

To gain some insight into the heterogeneous plate theory established in Section 2 the uni-axial frequency equation of an infinite two-layer isotropic plate is derived.

The system (130)–(132) with $F_x = 0$, is first transformed to a single sixth-order equation in terms of w^0 in the form

$$H_1 w^0 + H_2 p_z = 0 \tag{150}$$

where

$$\begin{aligned} H_1 = {} & (A_{11}D_{11} - B_{11}^2)(\ )_{,xxxxxx} + (2B_{11}R_1 - A_{11}R_2 - D_{11}R_0 \\ & + [B_{11}^2 R_0 - A_{11}D_{11}]R_0/A_{55})(\ )_{,xxxxtt} \\ & + (R_0R_2 - R_1^2 + [A_{11}R_2 + D_{11}R_0 - 2B_{11}R_1]R_0/A_{55})(\ )_{,xxtttt} \\ & + ([R_1^2 - R_0R_2]R_0/A_{55})(\ )_{,tttttt} + A_{11}R_0(\ )_{,xxtt} - R_0^2(\ )_{,tttt}. \end{aligned} \tag{151}$$

$$\begin{aligned} H_2 = {} & ([A_{11}D_{11} - B_{11}^2]/A_{55})(\ )_{,xxxx} + ([2B_{11}R_1 - A_{11}R_2 - D_{11}R_0]/A_{55})(\ )_{,xxtt} \\ & + ([R_0R_2 - R_1^2]/A_{55})(\ )_{,tttt} - A_{11}(\ )_{,xx} + R_0(\ )_{,tt}. \end{aligned} \tag{152}$$

If the x–y plane is located at the interface of the top layer "1" (of thickness "a") and the bottom layer "2" (of thickness "b"), having elastic moduli $\lambda_i, \mu_i, \nu_i (i = 1, 2)$ one finds that

$$A_{11} = 2\mu_1 a/(1-\nu_1) + 2\mu_2 b/(1-\nu_2), \qquad A_{55} = \mu_1 a + \mu_2 b \tag{153a, b}$$

$$B_{11} = \mu_1 a^2/(1-\nu_1) - \mu_2 b^2/(1-\nu_2) \tag{154}$$

$$D_{11} = 2\mu_1 a^3/3(1-\nu_1) + 2\mu_2 b^3/3(1-\nu_2) \tag{155}$$

$$R_0 = a\rho_1 + b\rho_2, \qquad R_1 = a^2\rho_1/2 - b^2\rho_2/2, \qquad R_3 = a^3\rho_1/3 + b^3\rho_2/3. \tag{156a, b, c}$$

The corresponding Kirchhoff's type theory in terms of a single sixth-order equation in w^0 takes the following form after making use of equations (143) and (144).

$$H_3 w^0 + H_4 p_z = 0 \tag{157}$$

where

$$\begin{aligned} H_3 = {} & (A_{11}D_{11} - B_{11}^2)(\ )_{,xxxxxx} + (2B_{11}R_1 - A_{11}R_2 - D_{11}R_0)(\ )_{,xxxxtt} \\ & + (R_0R_2 - R_1^2)(\ )_{,xxtttt} - A_{11}R_0(\ )_{,xxtt} + R_0^2(\ )_{,tttt} \end{aligned} \tag{158}$$

$$H_4 = R_0(\ )_{,tt} - A_{11}(\ )_{,xx}. \tag{159}$$

The frequency equation is derived from the equation of motion (150) or (157) by dropping the p_z term and by introducing for the transverse deflection

$$w^0 = \exp i(pt + fx) \tag{160}$$

where p is the circular frequency, f is the wave number related to the wavelength Λ through

$$f\Lambda = 2\pi. \tag{161}$$

The frequency equation obtained from equation (150) is

$$c^6 + (K_3 - K_6/f^2)/K_4 \,.\, c^4 + (K_2 - K_5/f^2)/K_4 \,.\, c^2 + K_1/K_4 = 0 \tag{162}$$

while the corresponding equation based on Kirchhoff's theory (157) is

$$(K_3'-K_6/f^2)c^4+(K_2'-K_5/f^2)c^2+K_1 = 0 \tag{163}$$

where

$$K_1 = A_{11}D_{11}-B_{11}^2, \qquad K_2 = K_2'+(B_{11}^2R_0-A_{11}D_{11})R_0/A_{55} \tag{164a, b}$$

$$K_3 = K_3'+(A_{11}R_2+D_{11}R_0-2B_{11}R_1)R_0/A_{55}, K_4 = (R_1^2-R_0R_2)R_0/A_{55} \tag{164c, d}$$

$$K_5 = A_{11}R_0, \qquad K_6 = -R_0^2, \qquad K_2' = 2B_{11}R_1-A_{11}R_2-D_{11}R_0,$$
$$K_3' = R_0R_2-R_1^2 \tag{164e-h}$$

and

$$c = p/f. \tag{165}$$

For a given plate the frequency equation (162) or (163) can be used to determine c for a specified value of f or the wavelength Λ. In other words, frequency curves (p vs. f) can be determined without difficulty by the suggested plate theories. Before going into a numerical example, the three-dimensional elasticity solution is derived for the plane wave propagation problem in an infinite two-layer isotropic plate.

8. ELASTICITY SOLUTION FOR WAVE PROPAGATION IN TWO-LAYER PLATES

The frequency equation for the wave propagation in an infinite two-layer isotropic plate in plane strain was derived by the authors and independently by Jones [11].

One finds the following non-dimensional transcendental frequency equation

$$\begin{aligned} &a_1+a_2\,\mathrm{sh}\alpha_2 b\,\mathrm{sh}\beta_2 b+a_3\,\mathrm{ch}\alpha_2 b\,\mathrm{ch}\beta_2 b+\mathrm{sh}\alpha_1 a\,\mathrm{sh}\beta_1 a(a_6+a_4\,\mathrm{sh}\alpha_2 b\,\mathrm{sh}\beta_2 b\\ &+a_5\,\mathrm{ch}\alpha_2 b\,\mathrm{ch}\beta_2 b)+\mathrm{ch}\alpha_1 a\,\mathrm{ch}\beta_1 a(a_9+a_7\,\mathrm{sh}\alpha_2 b\,\mathrm{sh}\beta_2 b+a_8\,\mathrm{ch}\alpha_2 b\,\mathrm{ch}\beta_2 b)\\ &+\mathrm{sh}\alpha_1 a\,\mathrm{ch}\beta_1 a(a_{10}\mathrm{sh}\alpha_2 b\,\mathrm{ch}\beta_2 b+a_{11}\mathrm{ch}\alpha_2 b\,\mathrm{sh}\beta_2 b)+\mathrm{ch}\alpha_1 a\,\mathrm{sh}\beta_1 a(a_{12}\,\mathrm{sh}\alpha_2 b\\ &\mathrm{ch}\beta_2 b+a_{13}\,\mathrm{ch}\alpha_2 b\,\mathrm{sh}\beta_2 b) = 0 \end{aligned} \tag{166}$$

where the appropriate constants are given in the Appendix.

For a given plate all parameters and constants are easily computed. Then the frequency equation (166) is used to determine the wave number F for an assumed non-dimensional velocity Ω by a laborious trial and error procedure.

Equation (166) has many solutions for a given Ω which means that the frequency curves have many branches. Each branch corresponds to a particular wave in the plate with a definite amplitude distribution across its thickness.

Note that for a homogeneous plate one has $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, $a = b$, $\mu_1 = \mu_2$ and equation (166) reduces to two frequency equations of symmetric and antisymmetric motions as given by Lamb [7].

Another interesting reduction of equation (166) occurs when waves of very small length are considered, one finds the velocity of the Rayleigh surface waves for each layer.

Setting $f = 0$ in equation (166) the following cut-off frequency equations are obtained for simple thickness-vibrations, which are limiting forms of waves in a plate as the wave length approaches infinity,

$$\beta_1 = 0 \tag{167}$$

$$(\alpha_1/\alpha_2)\beta_2^2 \sin \alpha_2 b \cos \alpha_1 a + (\mu_1/\mu_2)\beta_1^2 \sin \alpha_1 a \cos \alpha_2 b = 0 \tag{168}$$

$$\beta_2 \sin \beta_2 b \cos \beta_1 a + (\mu_1/\mu_2)\beta_1 \sin \beta_1 a \cos \beta_2 b = 0 \tag{169}$$

where

$$\alpha_i = p^2\rho_i/(\lambda_i + 2\mu_i), \qquad \beta_i = p^2\rho_i/\mu_i \qquad i = 1, 2 \tag{170a, b}$$

Equation (169) simply gives zero circular frequency p. Equations (168), (169) are the frequency equations for the simple thickness-stretch and simple thickness-shear modes, respectively, of a laminated plate.

By definition, in a free thickness-stretch mode of the plate $u = 0$, $w = w(z, t)$ and in a thickness-shear mode $u = u(z, t)$, $w = 0$. Consequently, these modes could be directly obtained from the equations of motion.

Since the solution of the three-dimensional elasticity equations is obtained for plane waves in two-layer plates, it becomes possible to examine the proposed plate theory.

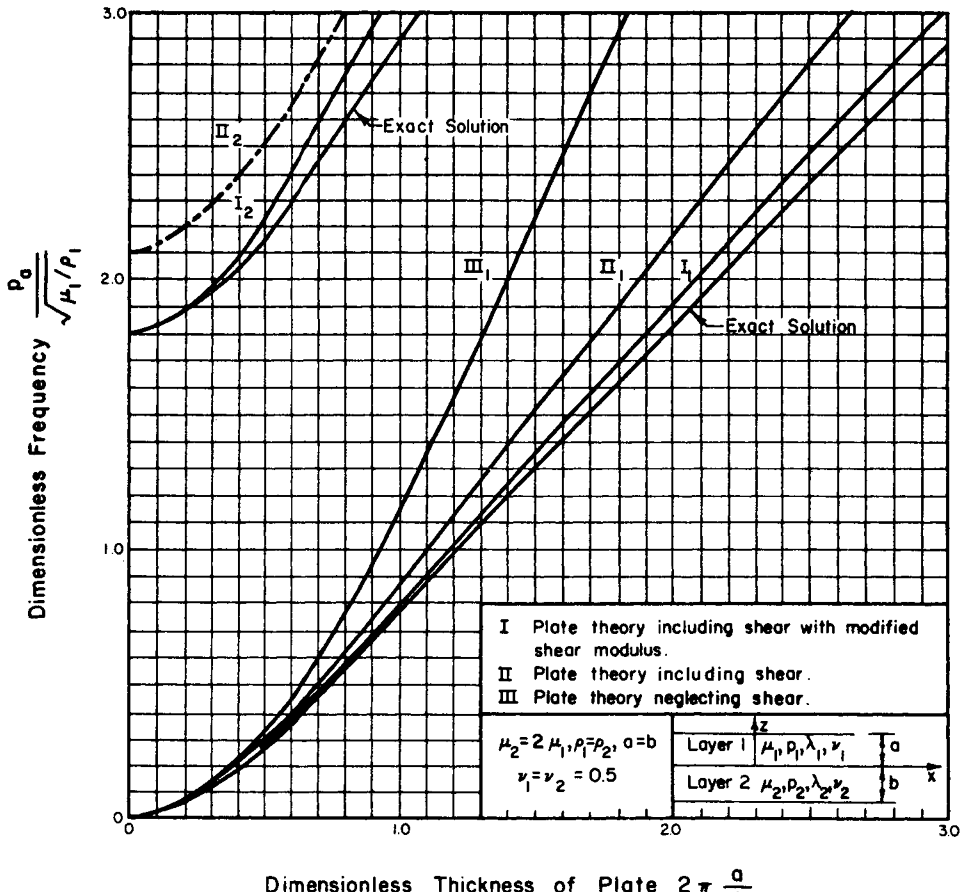

FIG. 1. Frequency curves of an infinite two-ply laminated plate according to various theories.

In the simple thickness-shear motion, equation (169) is the next frequency equation for the layered plate. The corresponding result based on the plate theory (143)–(145) is

$$p^2 = \bar{K}A_{55}/(R_2 - R_1^2/R_0) \tag{171}$$

where the value of $\bar{K}$ is so determined as to make the lowest root of p calculated from equation (169) equal to the approximate solution (171). The value of $\bar{K}$ will depend on the material properties of the layered plate whereas in homogeneous plate it has a constant value, (see Ref. [9], p. 37).

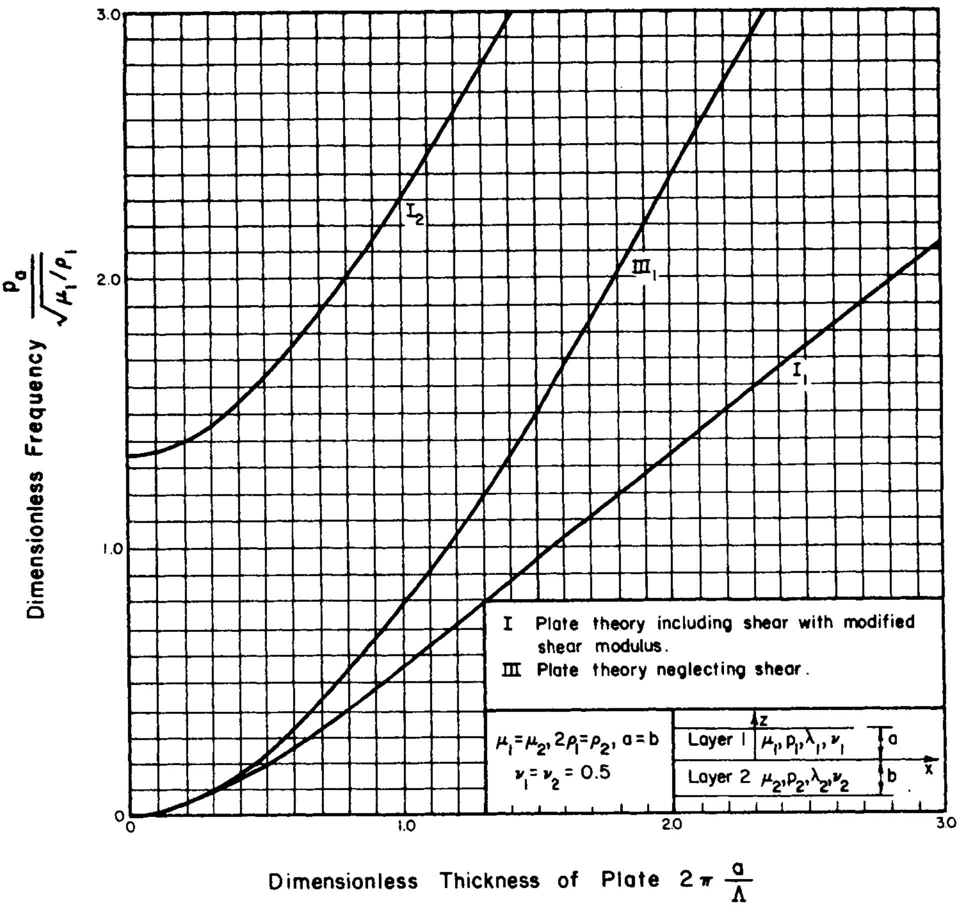

FIG. 2. Frequency curves of an infinite two-ply laminated plate according to plate theories.

9. NUMERICAL EXAMPLES

Frequency curves for two-layer isotropic plates of infinite length are obtained by the various theories developed in the previous sections as follows:

(i) The heterogeneous plate theory including transverse shear deformations with modified modulus $\bar{K}A_{55}$ instead of A_{55} for which the frequency equation is given by equation (150) and $\bar{K}$ is determined by equation (169) and (171).

(ii) The heterogeneous plate theory including transverse shear deformations for which the frequency equation is given by equation (150).

(iii) The heterogeneous plate theory neglecting transverse shear deformations for which the frequency equation is given by equation (157).

(iv) The exact solution from the elasticity theory for which the frequency equation is given by equation (166).

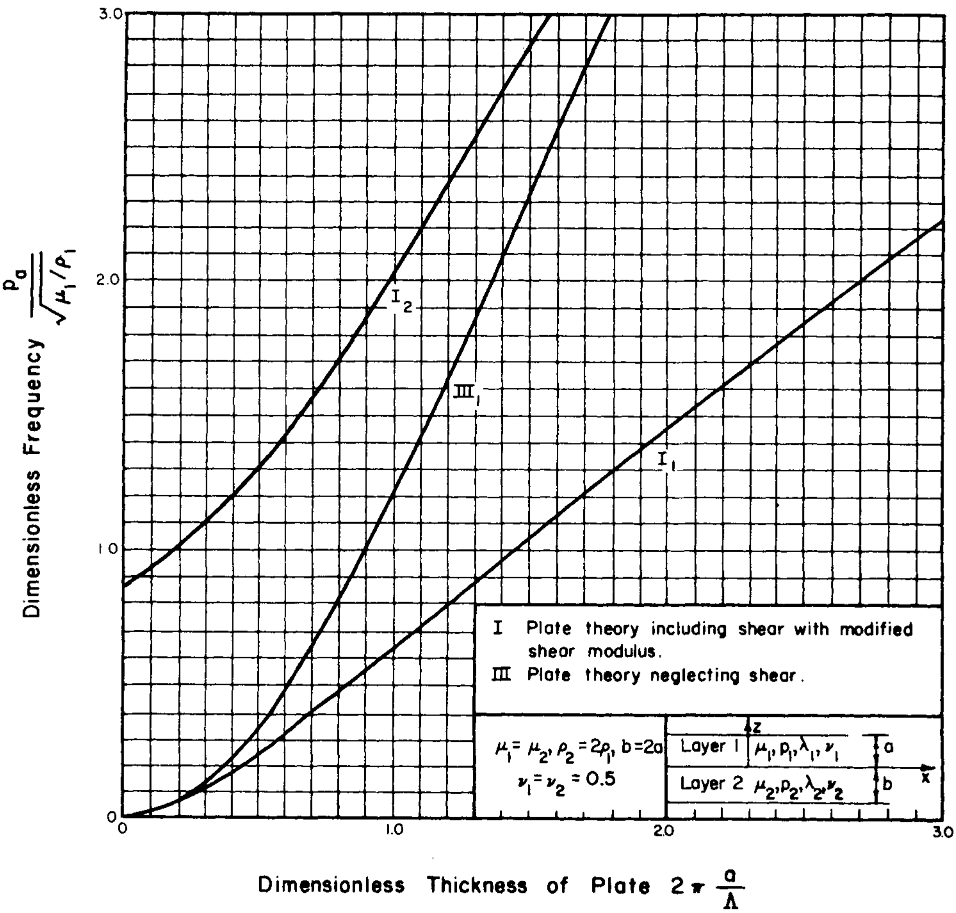

FIG. 3. Frequency curves of an infinite two-ply laminated plate according to plate theories.

In Fig. 1, two branches of the frequency curves of each of the above mentioned theories are presented for a two-layer plate with material properties $\mu_1 = \frac{1}{2}\mu_2, \rho_1 = \rho_2, \nu_1 = \nu_2$ and $a = b$. Those curves with subscripts 1 and 2 are, respectively, the lowest branch of all frequency curves and the branch of the first cut-off frequency. The first cut-off frequency is the lowest frequency of the simple thickness-shear mode. In the vibration of homogeneous plates the lowest branch of all frequency curves corresponds to the first anti-symmetric vibration or flexural vibration whereas the branch of the first cut-off frequency corresponds to another branch of the anti-symmetric vibration of higher frequency. The second lowest branch which corresponds to the longitudinal vibration in the case of homogeneous plates is not shown. It is noted that the heterogeneous plate theory including transverse shear deformations gives only three branches of the frequency curve in view of equation (150) while the exact solution has many branches. The three branches are the lowest, the second lowest and the first cut-off frequency branch. According to equation (157) the heterogeneous plate theory, neglecting shear deformations, gives only two branches and none of them is the branch of the first cut-off frequency. Thus, only one branch of the curve given by the plate theory neglecting shear deformations is shown in Fig. 1.

For two more two-layer isotropic plates, the frequency curves of the heterogeneous plate theories are shown in Figs. 2 and 3. In order to have a basis for comparison, the frequency curves of the plate theories for a homogeneous plate are presented in Fig. 4.

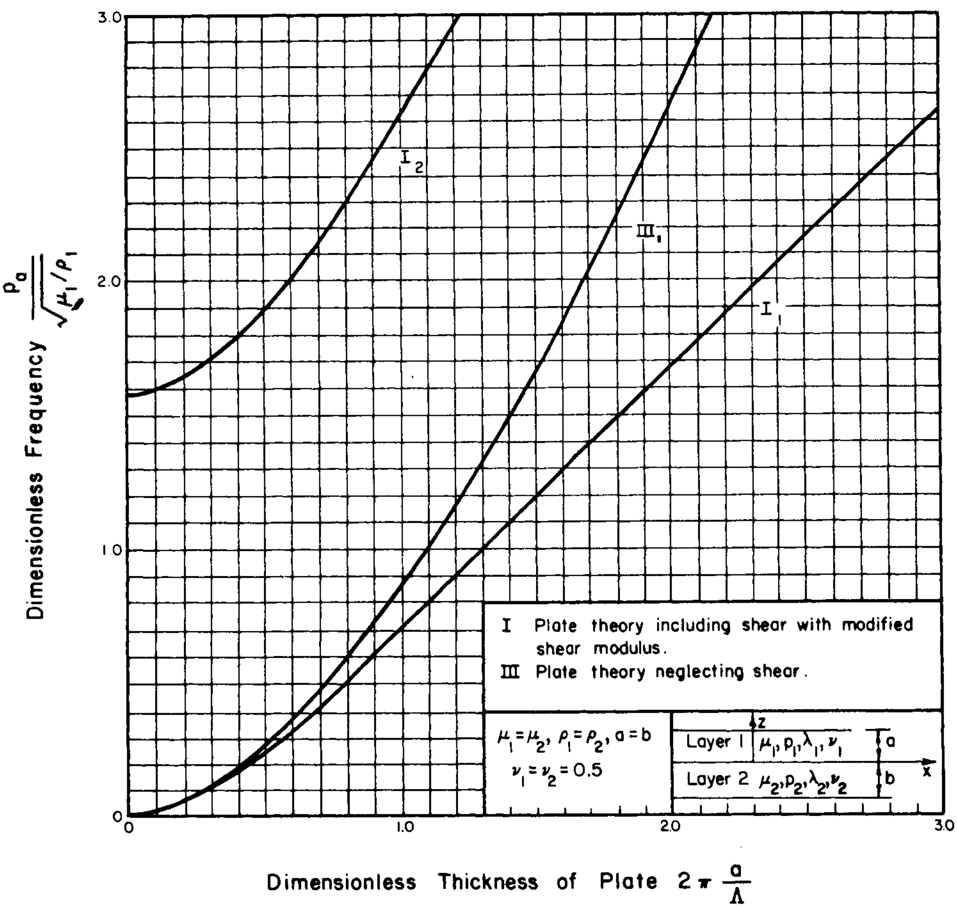

FIG. 4. Frequency curves of a homogeneous plate according to plate theories.

10. DISCUSSIONS AND CONCLUSIONS

Some interesting results of the above numerical examples are discussed and concluded as follows:

(1) The frequency curves presented in Fig. 1 for an infinite two-layer isotropic plate show good agreement between the predictions of the exact solution of elasticity and that of the heterogeneous plate theory including rotatory inertia and transverse shear deformations with a modified modulus $\bar{K}A_{55}$.

(2) The shear effect is important for heterogeneous plates in vibration. According to Figs. 1, 2, and 3, the frequency curve of the plate theory including shear (curve I_1) always deviated from that of the plate theory neglecting shear (curve III_1). Therefore, the effect of transverse shear deformations is significant in the dynamic theory of heterogeneous plates.

(3) The first cut-off frequency branch of the frequency curve is important in the vibration of heterogeneous plates. In Fig. 3, the cut-off frequency 0·87 is much lower and closer to the lowest branch for the specific two-layer plate than in the case of the homogeneous plate (Fig. 4) which has a value of $\pi/2$. Since the branch of the first cut-off frequency may be in the lower frequency region, the vibration of a heterogeneous plate may possibly be in the branch of the first cut-off frequency rather than in the lowest branch even when the vibration is of a rather low frequency. Furthermore, in view of the present results, the plate theory including shear is adequate to describe the motion of a heterogeneous plate since this theory does provide the branch of the first cut-off frequency.

Acknowledgements—This paper is a result of an investigation that was sponsored by the Office of Naval Research. The use of M.I.T. computer facilities in this research is gratefully acknowledged.

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APPENDIX

The appropriate constants of the non-dimensional transcendental frequency equation (166) are:

$$a_1 = 8Z_1Z_2(VX-2WY)$$

$$a_2 = 4Z_1[8Z_{11}WX-(Z_2^2/Z_{22})VY]$$

$$a_3 = 4Z_1(4VY-2Z_2^2WX)$$

$$a_4 = 4(Z_{11}/Z_{22})[Z_1^2X^2+16Z_{11}^2W^2]+(Z_2^2/Z_{11}Z_{22})[Z_1^2Y^2+4Z_{11}^2V^2]$$

$$a_5 = -Z_2^2[(Z_1^2/Z_{11})X^2+16Z_{11}W^2]-4[(Z_1^2/Z_{11})Y^2+4Z_{11}V^2]$$

$$
\begin{aligned}
a_6 &= 4Z_2[8Z_{11}VW-(Z_1^2/Z_{11})XY] \\
a_7 &= -16Z_{22}(Z_1^2W^2+X^2)-(Z_2^2/Z_{22})(Z_1^2V^2+4Y^2) \\
a_8 &= 4Z_2(Z_1^2W^2+X^2)+4(Z_1^2V^2+4Y^2) \\
a_9 &= 8Z_2^2(2XY-Z_1^2VW) \\
a_{10} &= -4(VX+2WY)(Z_1^2/Z_{43}+Z_{43}Z_2^2) \\
a_{11} &= (VX+2WY)(Z_1^2Z_2^2+16Z_{12}^2)/Z_{12} \\
a_{12} &= (VX+2WY)(Z_1^2Z_2^2+16Z_{21}^2)/Z_{21} \\
a_{13} &= -4(VX+2WY)(Z_{34}Z_2^2+Z_1^2/Z_{34}).
\end{aligned}
\tag{A1-13}
$$

Introducing the wave parameters

$$
\begin{aligned}
&\alpha_i^2 = f^2-p^2\rho_i/(\lambda_i+2\mu_i), \qquad \beta_i^2 = f^2-\rho^2\rho_i/\mu_i \qquad i = 1,2 \\
&\Omega^2 = c^2\rho_1/\mu_1, \qquad F = fa = 2\pi a/\Lambda
\end{aligned}
\tag{A14-17}
$$

and the plate parameters

$$
\begin{aligned}
&\gamma = \rho_1/\rho_2, \qquad \delta = \mu_1/\mu_2, \qquad e = b/a \\
&k_i = \mu_i/(\lambda_i+2\mu_i) \qquad i = 1,2
\end{aligned}
\tag{A18-21}
$$

Then the constants V, W, X, Y, Z take the form

$$
\begin{aligned}
V &= 2(\delta-1)+\delta\gamma\Omega^2 \\
W &= \delta-1 \\
X &= \delta(2-\Omega^2)-2 \\
Y &= 2(1-\delta)+\Omega^2\delta(1-\gamma) \\
Z_1 &= 2-\Omega^2 \\
Z_2 &= 2-\gamma\delta\Omega^2 \\
Z_{11} &= [(1-k_1\Omega^2)(1-\Omega^2)]^{\frac{1}{2}} \\
Z_{22} &= [(1-k_2\gamma\delta\Omega^2)(1-\gamma\delta\Omega^2)]^{\frac{1}{2}} \\
Z_{12} &= [(1-k_1\Omega^2)(1-\gamma\delta\Omega^2)]^{\frac{1}{2}} \\
Z_{21} &= [(1-k_2\gamma\delta\Omega^2)(1-\Omega^2)]^{\frac{1}{2}} \\
Z_{34} &= [(1-\Omega^2)(1-\gamma\delta\Omega^2)]^{\frac{1}{2}} \\
Z_{43} &= [(1-k_1\Omega^2)(1-k_2\gamma\delta\Omega^2)]^{\frac{1}{2}}
\end{aligned}
\tag{A22-33}
$$

and the amplitudes of the hyperbolic functions are

$$
\begin{aligned}
\alpha_1 a &= F\sqrt{(1-k_1\Omega^2)} \\
\alpha_2 b &= Fe\sqrt{(1-k_2\gamma\delta\Omega^2)} \\
\beta_1 a &= F\sqrt{(1-\Omega^2)} \\
\beta_2 b &= Fe\sqrt{(1-\gamma\delta\Omega^2)}
\end{aligned}
\tag{A34-37}
$$

(*Received* 16 *September* 1965)

Résumé—Une théorie de mouvements linéaires, à deux dimensions, de plaques hétérogènes, est déduite de la théorie à trois dimensions d'élasticité. Des déformations transversales de cisaillement et d'inertie rotatoire sont inclues dans cette théorie générale. L'hétérogénéité du matériel est considérée se trouver seulement dans la direction de l'épaisseur de la plaque. La théorie générale de plaques se spécialise dans les cas de plaques aélotropes, orthotropes et isotropes symmétriquement laminées. Une théorie de plaques du type Kirchhoff est aussi déduite. Des équations de fréquence pour la propagation des ondes harmoniques dans une plaque infinie isotrope à deux couches en contrainte plane sont obtenues par l'élasticité et la présente théorie de plaques. Plusieurs exemples numériques sont résolus et leurs résultats sont présentés en graphiques.

Zusammenfassung—Eine zweidimensionale lineare Theorie von Bewegungen heterogener Platten ist von der dreidimensionalen Elastizitätstheorie abgeleitet. Transversale Schubbeanspruchungsverformungen und Drehungsträgheit sind in der gegenwärtigen allgemeinen Theorie eingeschlossen. Die Heterogenität des Materials ist nur in der Dickenrichtung der Platte erwogen. Die allgemeine Plattentheorie ist für Fälle von symmetrisch geschichteten aeolotropischen, orthotropischen und isotropischen Platten spezialisiert. Eine Plattentheorie der Kirchhoff Type ist ebenfalls abgeleitet. Frequenzgleichungen für die Fortpflanzung von harminischen Wellen in einer unendlichen zweischichtigen isotropischen Platte für ebenc Spannunszustände werden von der elastischen und der gegenwärtigen Plattentheorie erhalten. Einige zahlenmässige Beispiele und deren Ergebnisse sind in graphischer Darstellung wiedergegeben.

Абстракт—Двумерная линейная теория движений гетерогенных пластин выводится из трёхмерной теории упругости. Деформации поперечного сдвига и вращательная инерция включены в настоящую общую теорию. Считается, что гетерогенность материала существует только в направлении толщины пластины. Общая теория пластины упрощается случаям симметрично слоистых, олотропных, ортотропных и изотропных пластин. Выводится также теория пластины Кирхгоффа. Уравнения частоты для распространения гармонических волн в безграничной двуслойной изотропической пластине в плоской деформации получены теориями упругости и настоящей теорией пластины. Разрешено несколько цифровых примеров и их результаты представлены в диаграммах.